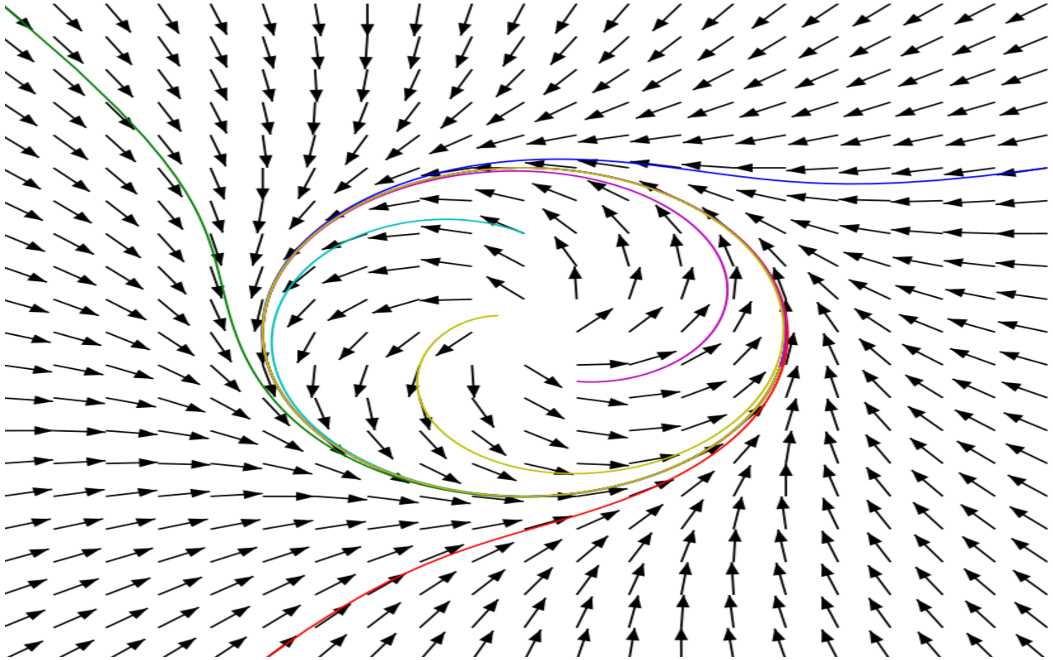
Equilibrium Cycle:
A "dynamic" equilibrium & its application in ride hailing



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(joint work with Tushar Walunj (IITB), Shiksha Singhal (LAAS-CNRS), Veeraruna Kavitha (IITB)

Nash equilibrium:

- Cornerstone of non-cooperative game theory
- Equilibrium of game dynamics (e.g., best response)

But game dynamics need not converge!

Q: How to capture outcome of oscillatory game dynamics?

This talk

- Equilibrium Cycle

  A novel set-valued equilibrium notion that captures the outcome of oscillatory game dynamics

  [arXiv:2411.08471]

- An application in ride hailing

  [Dynamic games & Applications, 2025]

  [Allerton 2022]

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Example: A visibility game

Players $\mathscr{N} = \{b, r\}$

Actions $\mathscr{A}_i = [0, 1]$.

Utilities
$$U_b(a_b, a_r) = \begin{cases} a_r - a_b, & \text{if } a_b < a_r, \\ 0, & \text{if } a_b = a_r, \\ 1 - a_b, & \text{otherwise.} \end{cases}$$

Ref: (Hendricks et al. (1988), Lotker et al. (2008))

Example: A visibility game

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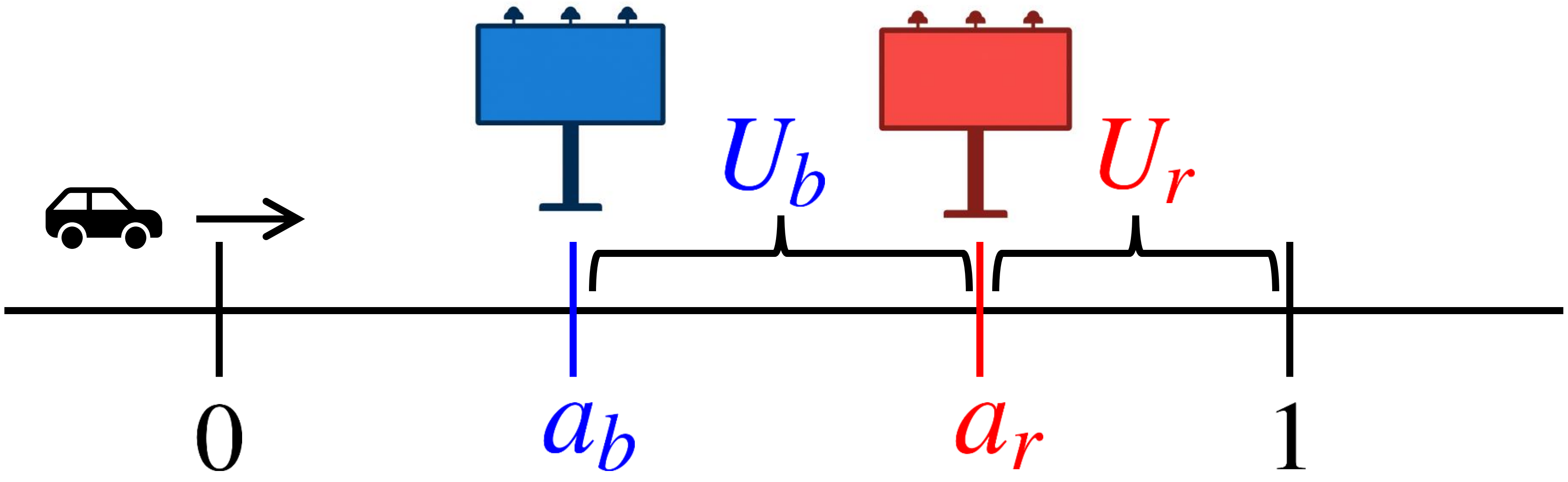
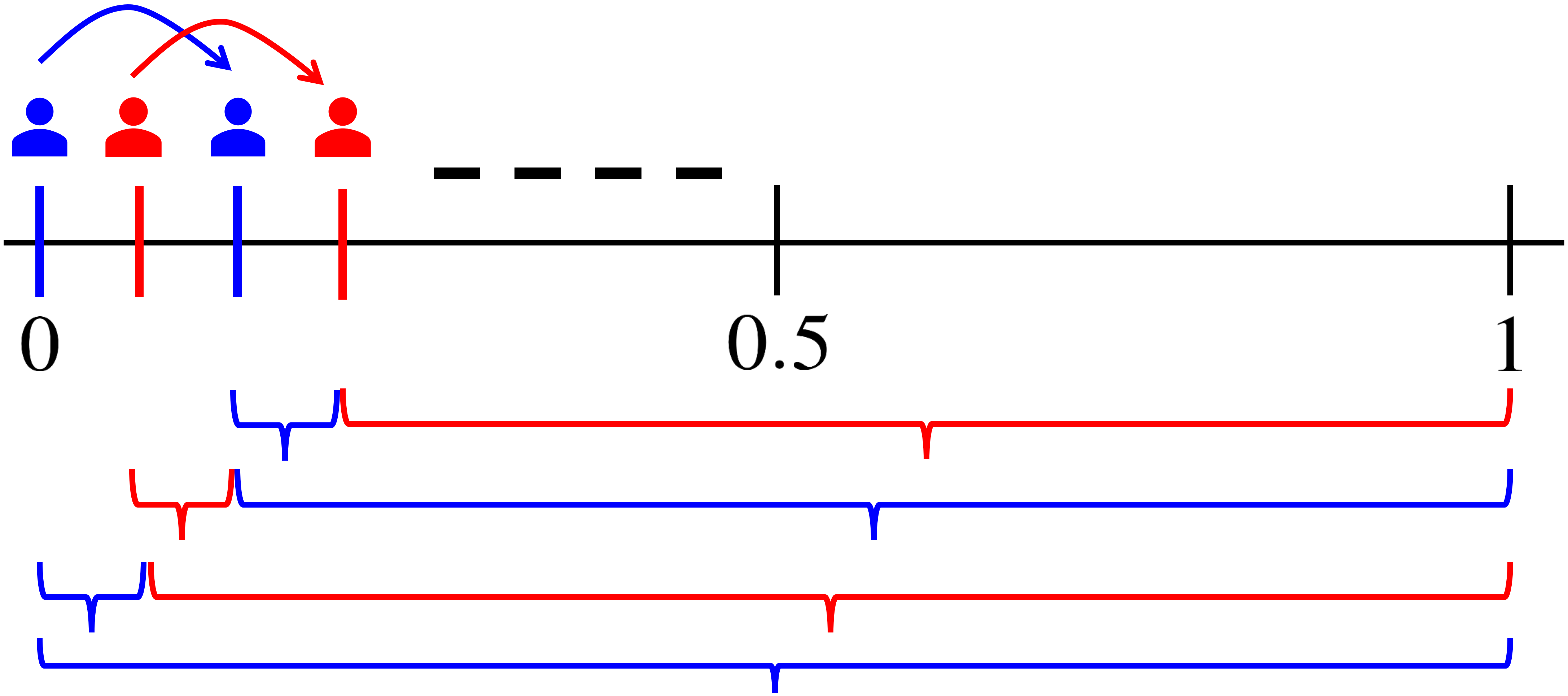


Example: A visibility game



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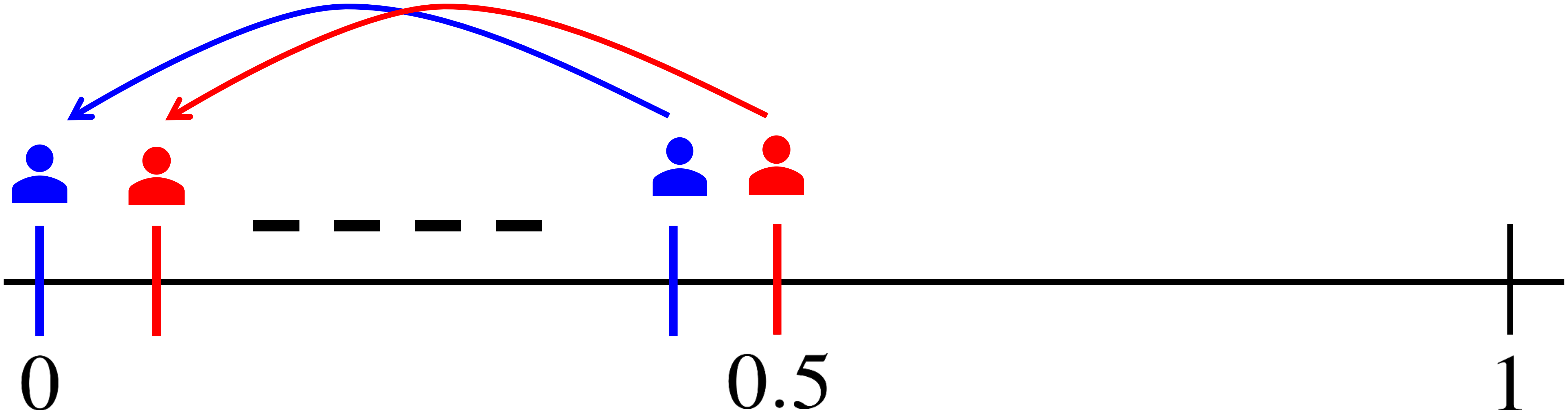
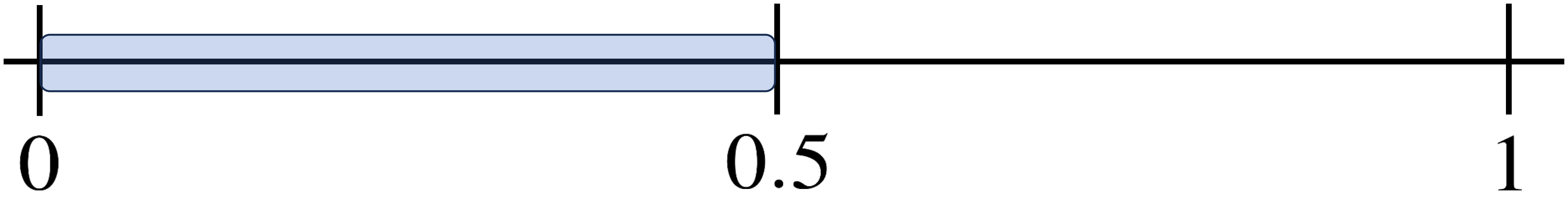


And they cycled ever after ...

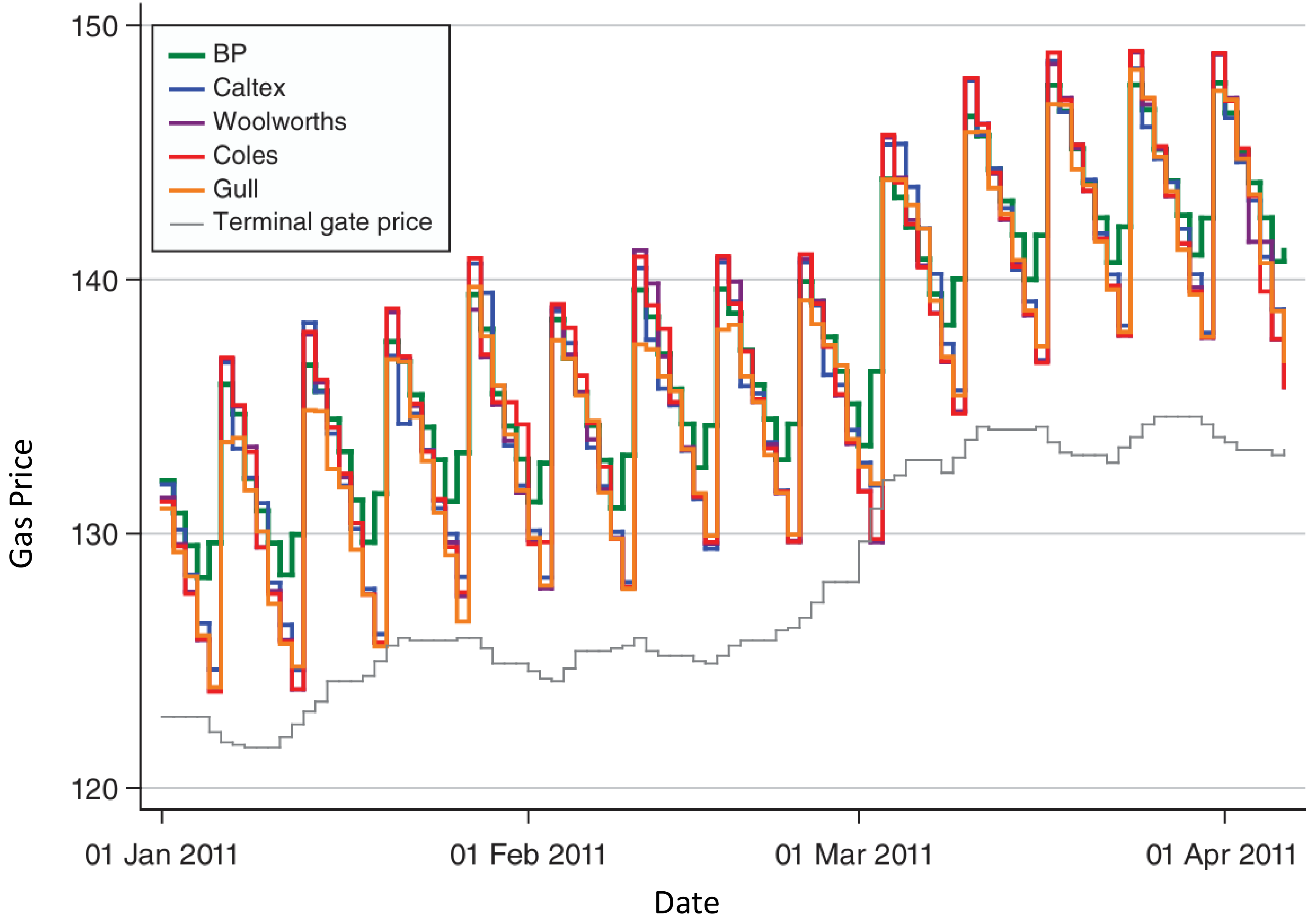
Example: A visibility game



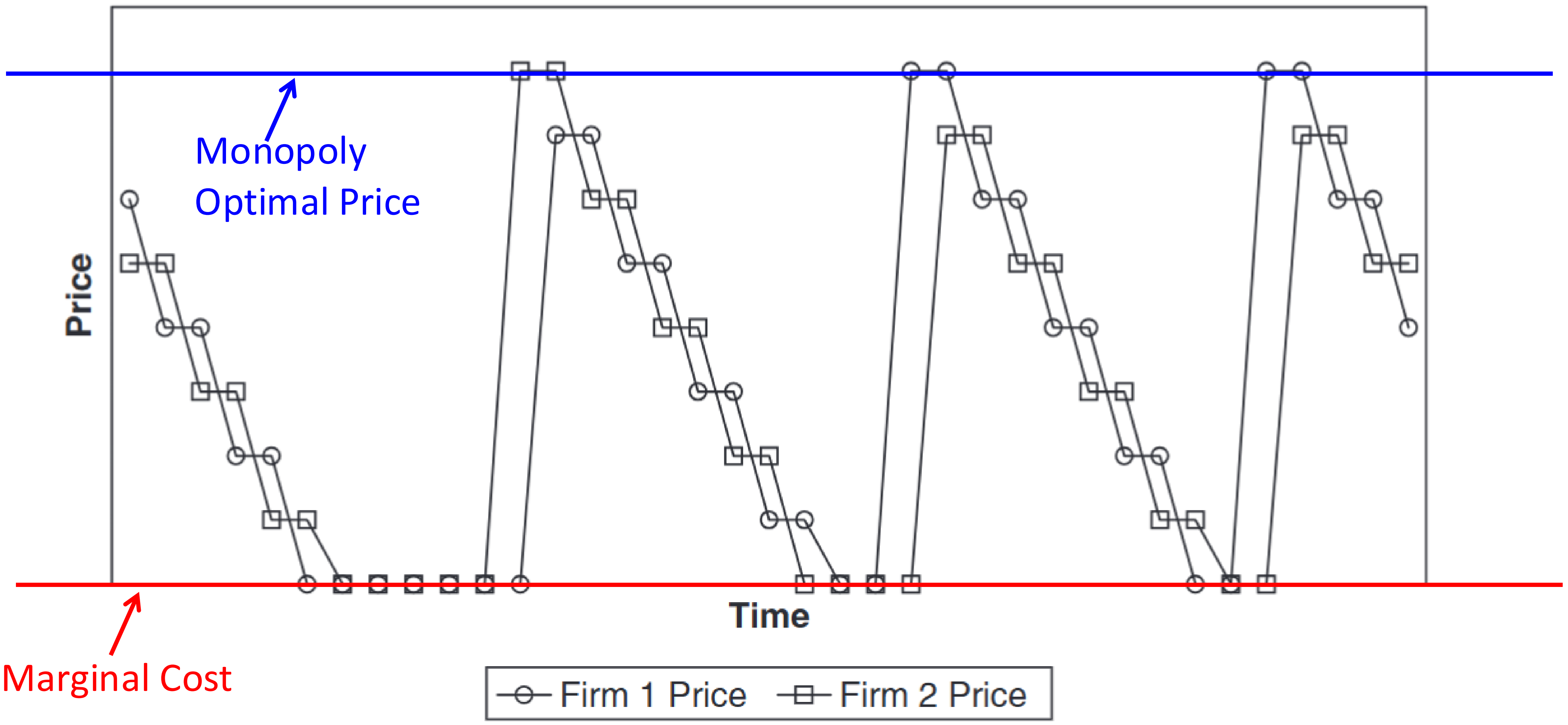
Dynamics oscillate over this interval

But (symmetric) mixed NE is supported over $[0, 1 - {}^{1}/_{e}]$

Ref: Lotker et al. (2008)



Ref: Byrne, David P., and Nicolas De Roos. "Learning to coordinate: A study in retail gasoline." *American Economic Review* 109.2 (2019): 591-619.

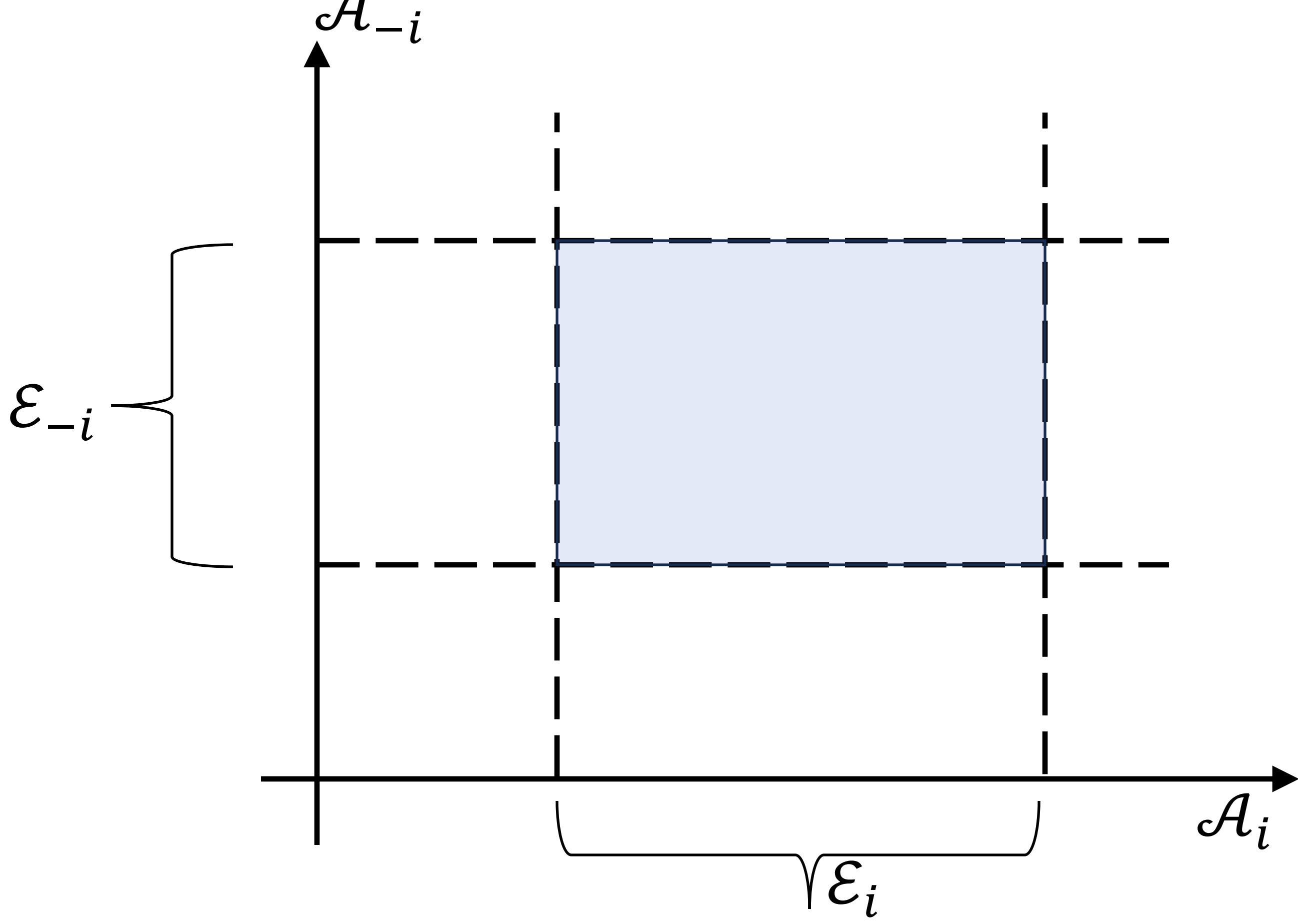


Ref: Noel, Michael D. "Edgeworth price cycles: Evidence from the Toronto retail gasoline market." *The Journal of Industrial Economics* 55.1 (2007)

Preliminaries

- $\mathcal{N} = \{1,2,\dots,N\}$ – finite set of players
- $\mathcal{A}_i$ - action space of player i (set in a metric space)
- $U_i\colon \mathcal{A} \to \mathbb{R}$ - utility function of player i (where $\mathcal{A} = \prod_{i\in\mathcal{N}} \mathcal{A}_i$)

A closed set $\mathcal{E} = \prod_{i \in \mathcal{N}} \mathcal{E}_i \subseteq \mathcal{A}$ is an **equilibrium cycle** if

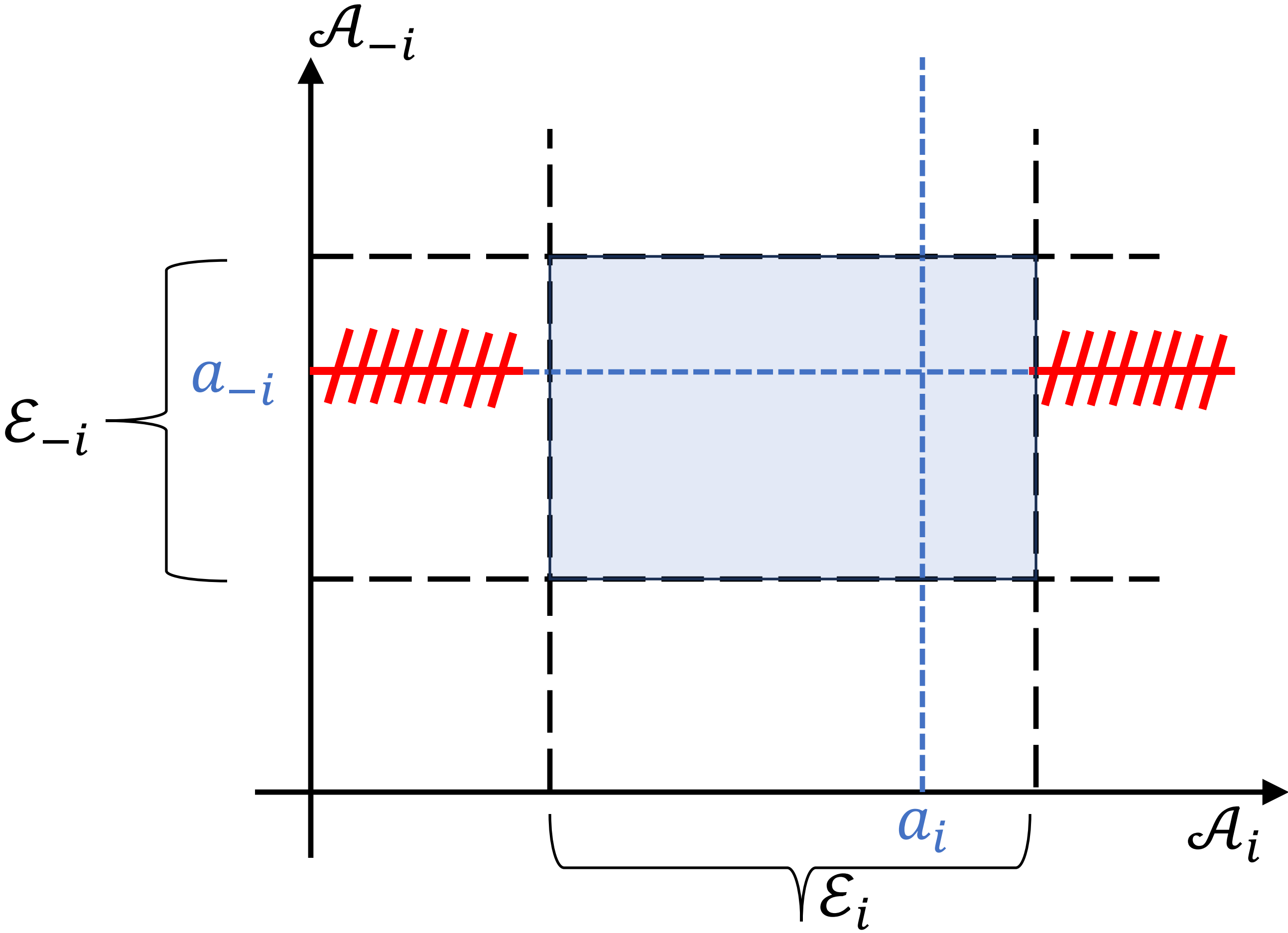


A closed set $\mathcal{E} = \prod_{i\in\mathcal{N}} \mathcal{E}_i \subseteq \mathcal{A}$ is an **equilibrium cycle** if

1. For any player i and $a_{-i} \in \mathcal{E}_{-i}$, there exists action $a_i \in \mathcal{E}_i$ such that

$$U_i(a_i, a_{-i}) > U_i(\tilde{a}_i, a_{-i}) \; \forall \; \tilde{a}_i \in \mathcal{A}_i \setminus \mathcal{E}_i$$

stability against external deviations



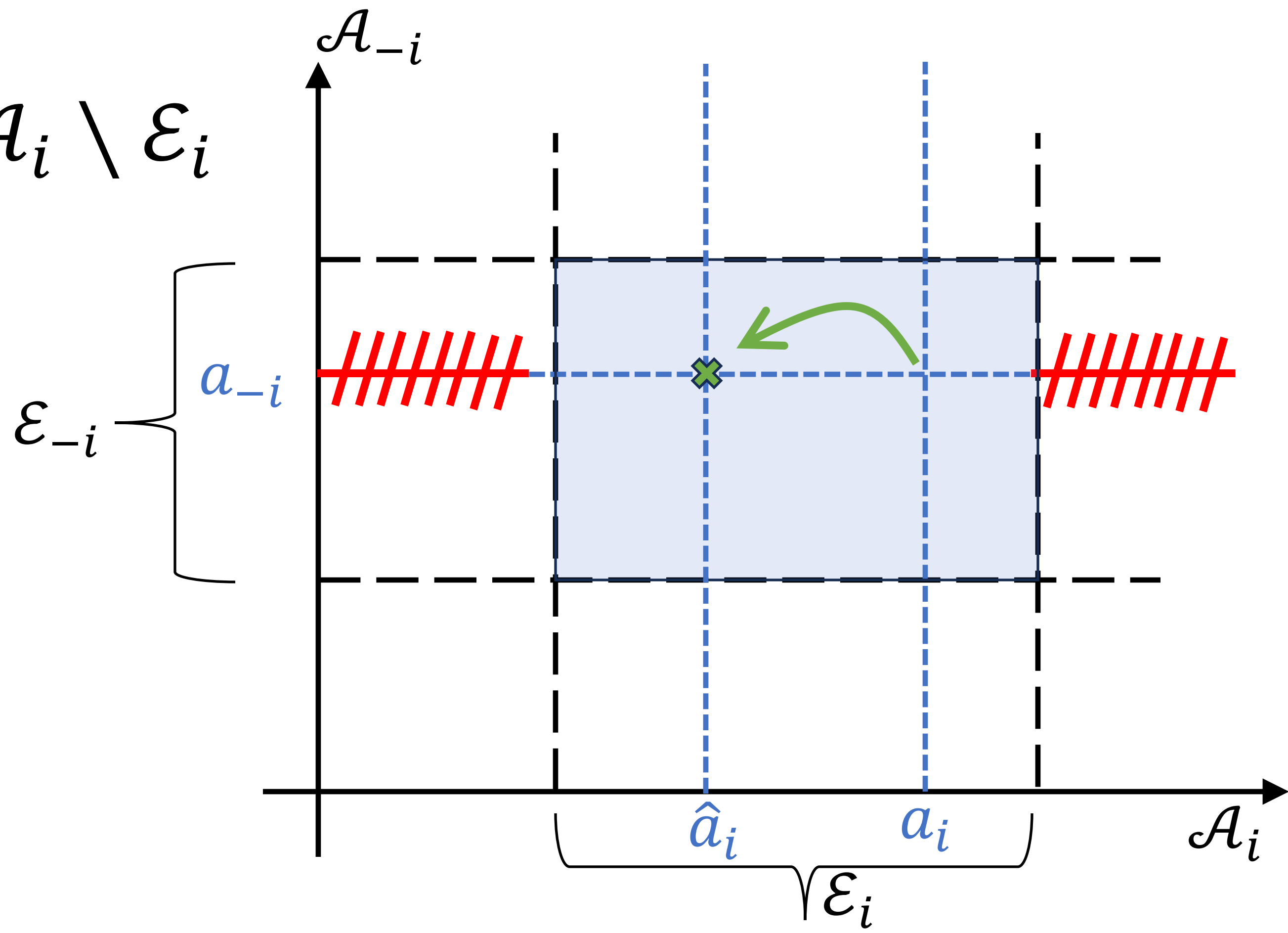
A closed set $\mathcal{E} = \prod_{i \in \mathcal{N}} \mathcal{E}_i \subseteq \mathcal{A}$ is an **equilibrium cycle** if

2. For any action profile a $\in \mathcal{E}$, there exists player i and an alternative action $\hat{a}_i \in \mathcal{E}_i$ such that:

$$U_i(\hat{a}_i, a_{-i}) > U_i(a_i, a_{-i})$$

$$U_i(\hat{a}_i, a_{-i}) > U_i(\tilde{a}_i, a_{-i}) \; \forall \; \tilde{a}_i \in \mathcal{A}_i \setminus \mathcal{E}_i$$

instability against internal deviations
"unrest within"

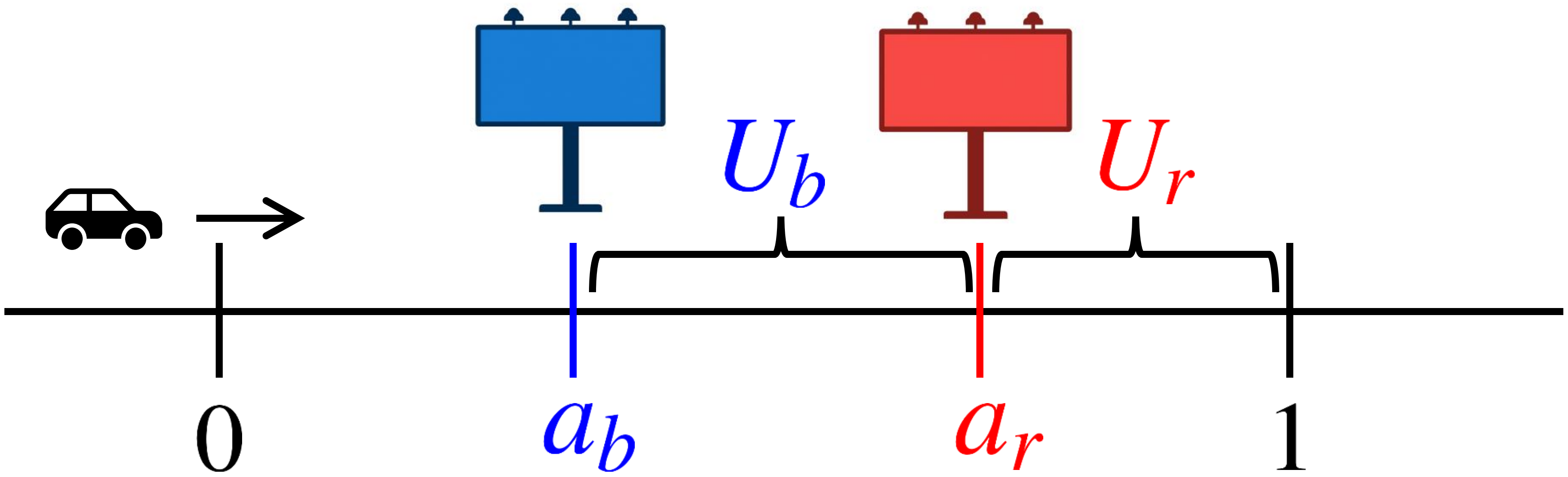


A closed set $\mathcal{E} = \prod_{i \in \mathcal{N}} \mathcal{E}_i \subseteq \mathcal{A}$ is an **equilibrium cycle** if

3. No strict subset of $\mathcal{E}$ satisfies the previous two conditions.

minimality

Revisiting the visibility game



Proposition: $\left[0, \frac{1}{2}\right]^2$ is an EC of this game

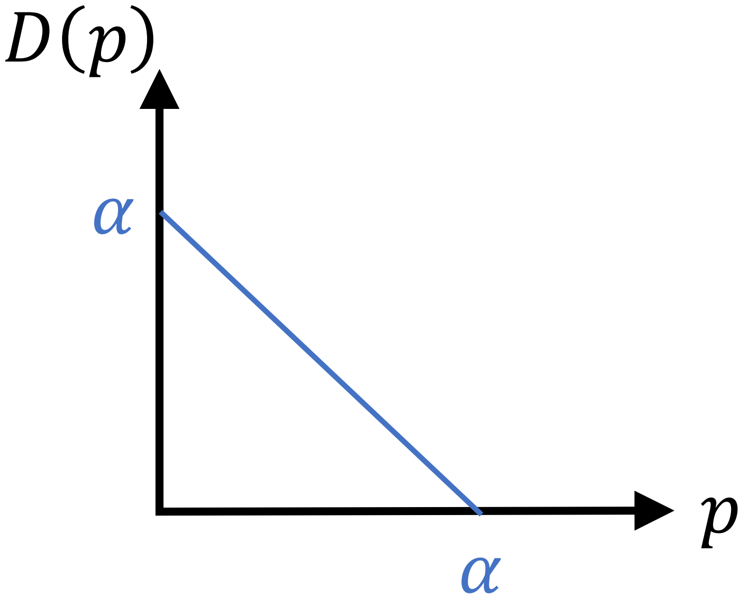
Ref: (Hendricks et al. (1988), Lotker et al. (2008))

Example: Bertrand duopoly

- Two producers/players
- Action space for each player i: Price $p_i \in [0, \infty)$

marginal cost of production

- $U_i(p_i, p_{-i}) = \begin{cases} (p_i - c)D(p_i) & \text{if } p_i < p_{-i} \\ & \end{cases}$



demand function

Example: Bertrand duopoly

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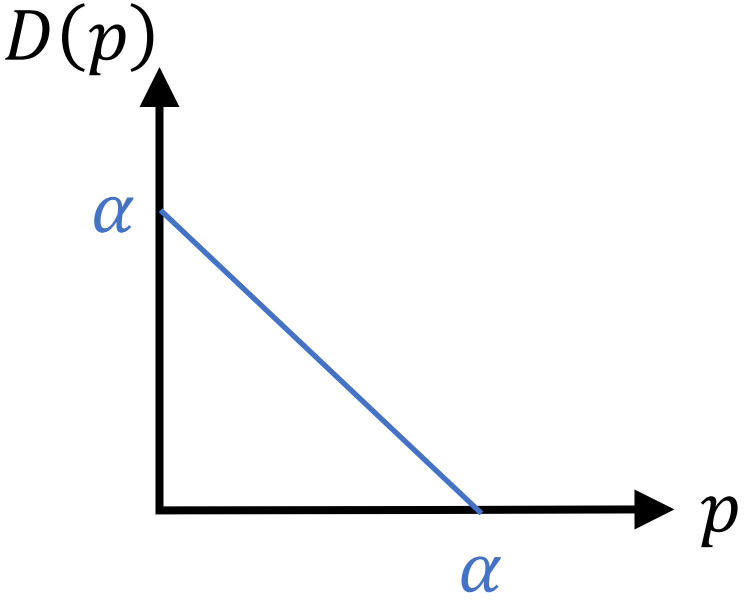
marginal cost of production

- $U_i(p_i, p_{-i}) = \begin{cases} (p_i - c)D(p_i) & \text{if } p_i < p_{-i} \\ \frac{1}{2}(p_i - c)D(p_i) & \text{if } p_i = p_{-i} \end{cases}$



demand function

Well known: (c, c) is the unique NE

marginal cost equilibrium

A twist: Bertrand duopoly with operating costs

not operating

- Action space for each player i: Price $p_i \in [0, \infty) \cup \{n_o\}$

operating cost

- $U_i(p_i, p_{-i}) = \begin{cases} (p_i - c)D(p_i) - O_c & \text{if } n_o \neq p_i < p_{-i} \\ & \\ & \end{cases}$

A twist: Bertrand duopoly with operating costs

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- Action space for each player i: Price $p_i \in [0, \infty) \cup \{n_o\}$

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Proposition: If $\alpha > c + 2\sqrt{O_c}$, then the game admits an EC.

condition for positive monopoly utility

Connection to classical equilibrium notions

- Curb set (Basu & Weibull (1991)): Subset of $\mathcal{A}$ that contains its best responses
- Minimal curb set: No subset is a curb set
- Non-trivial minimal curb set: Does not contain a pure NE

Player-2

Player-1

	L	C	R
U	(2, 0)	(0, 2)	(0, 0)
M	(0, 2)	(2, 0)	(0, 0)
D	(0, 0)	(0, 0)	(1, 1)

Two curb sets

Both are minimal

One one (orange) is non-trivial

Connection to classical equilibrium notions

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Theorem: Non-trivial minimal curb set $\Leftrightarrow$ Equilibrium Cycle

EC generalizes notion of curb sets beyond best response games (including discontinuous games)

Connection to classical equilibrium notions

- Finite game
- Best response graph: Nodes are action profiles
- $(a_i, a_{-i}) \to (\hat{a}_i, a_{-i})$ if $\hat{a}_i$ is a best response of player i against a_{-i}

- Note: A sink node of this graph is a pure NE

Connection to classical equilibrium notions

- Finite game
- Best response graph: Nodes are action profiles
- $(a_i, a_{-i}) \to (\hat{a}_i, a_{-i})$ if $\hat{a}_i$ is a best response of player i to a_{-i}

- Note: Random walks on this graph end up in a sink SCC

strongly connected component

Connection to classical equilibrium notions

- Finite game
- Best response graph: Nodes are action profiles
- $(a_i, a_{-i}) \to (\hat{a}_i, a_{-i})$ if $\hat{a}_i$ is a best response of player i to a_{-i}

Theorem:

- EC => sink SCC of best response graph
- Rectangular, non-singleton sink SCC of best response graph => EC

Equilibrium Cycle: Summary

- A *set-valued equilibrium concept* capturing ‘outcome’ of oscillatory game dynamics; seeks to capture *limit set* of these dynamics

- Key Feature:
  Applicable to discontinuous games where best responses may not exist

- Defining properties:
  1. **Stability** against external deviations
  2. **Instability** against internal deviations
  3. **Minimality**

- Connections:
  - Generalizes ‘minimal curb sets’ to discontinuous games
  - Related to ‘strongly connected sink components’ of BR graph in finite games

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  [arXiv:2411.08471]

- An application in ride hailing

  [Dynamic games & Applications, 2025]

  [Allerton 2022]

Matching platforms are everywhere

*logos subject to copyright

Ride hailing platforms

Considerable literature on:

- Modelling & performance evaluation
- Fleet sizing
- Optimal pricing, routing & fleet placement

Ride hailing platforms

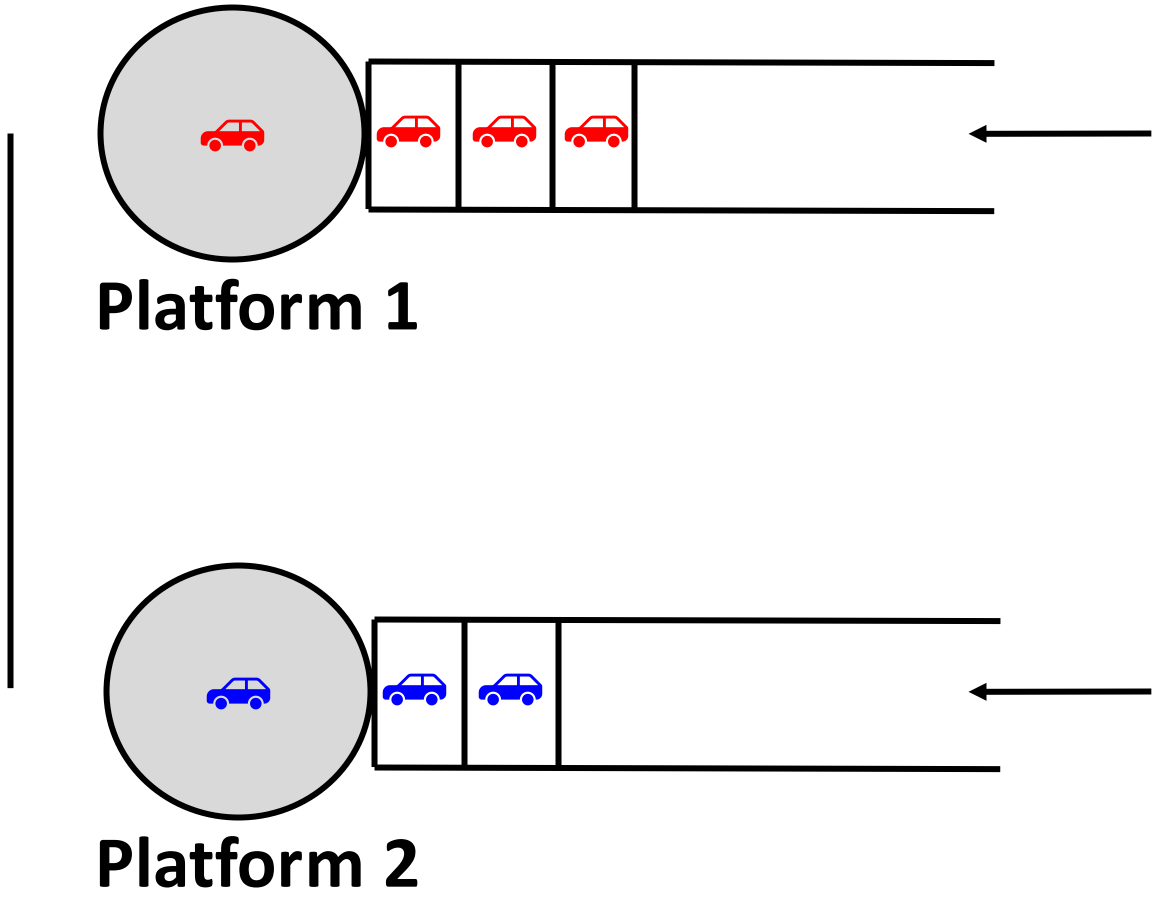
Uber

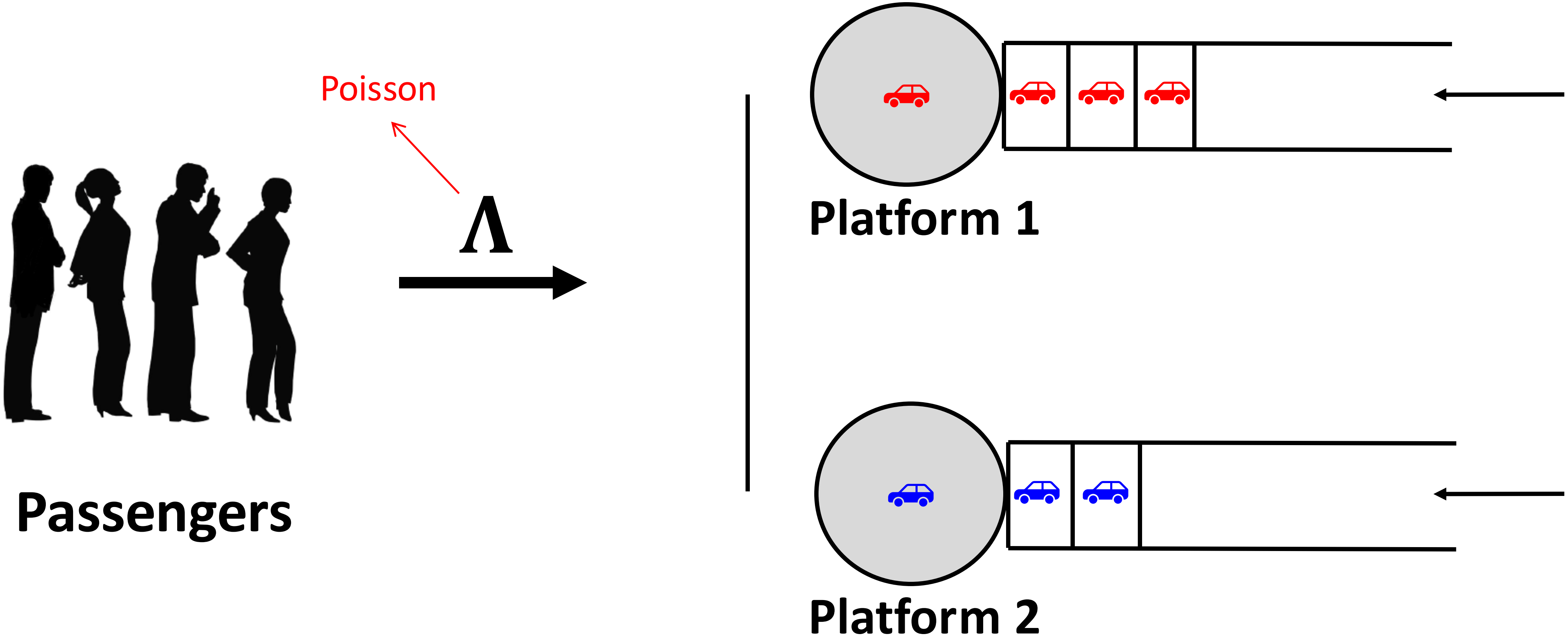
lyft

OLA

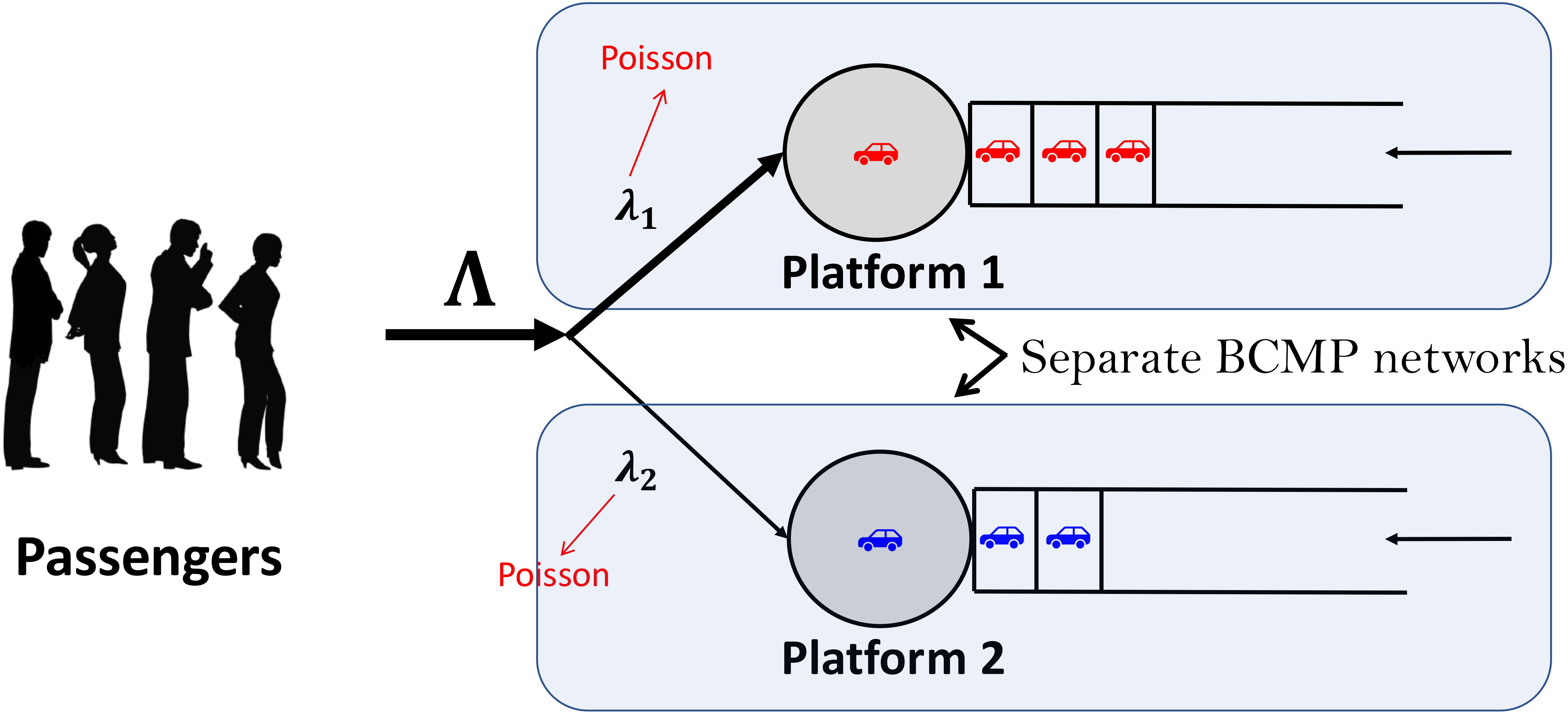
Q: What is the impact of ***competition*** between ride hailing platforms?

Two Competing platforms



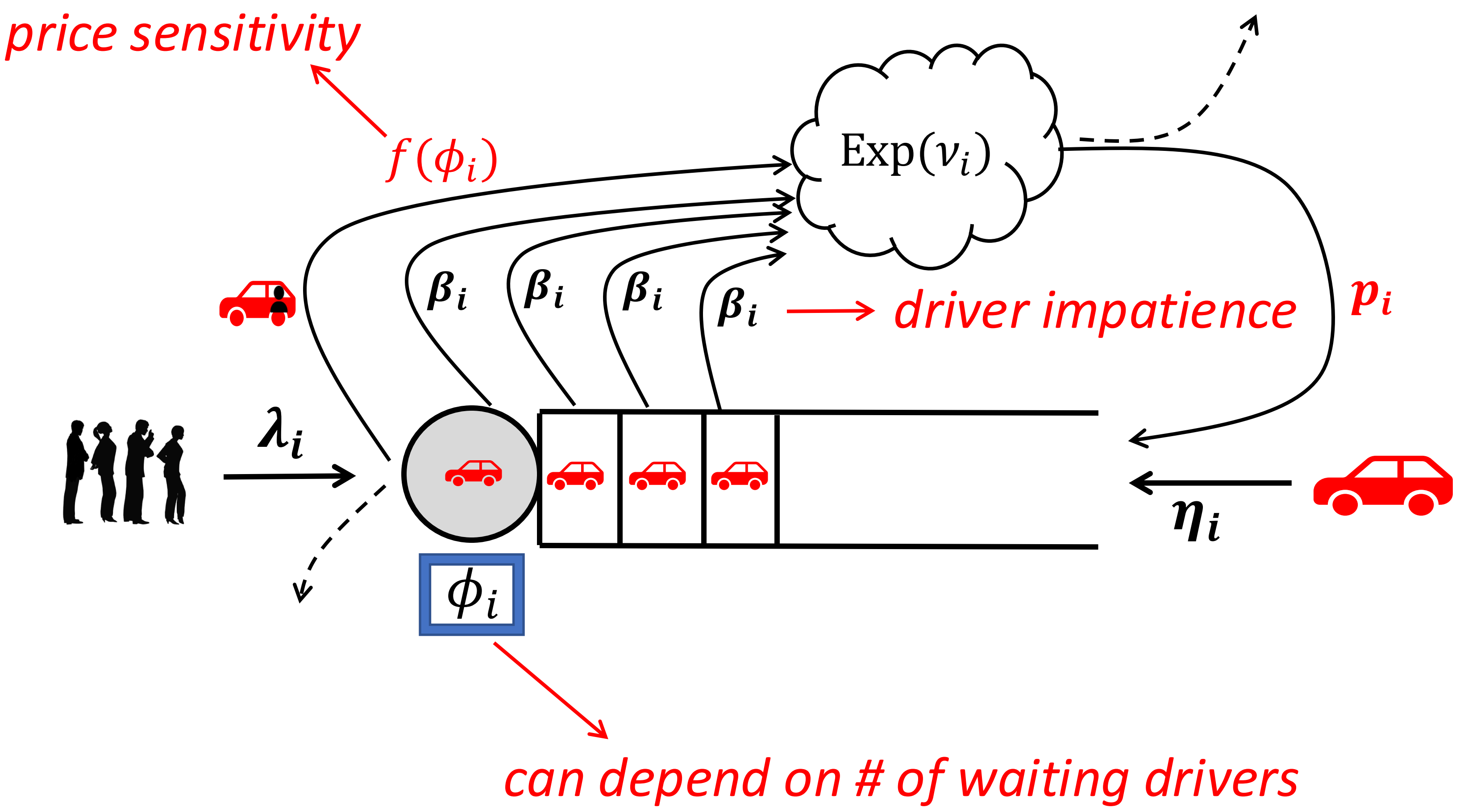


Poisson
Λ
Passengers
Platform 1
Platform 2

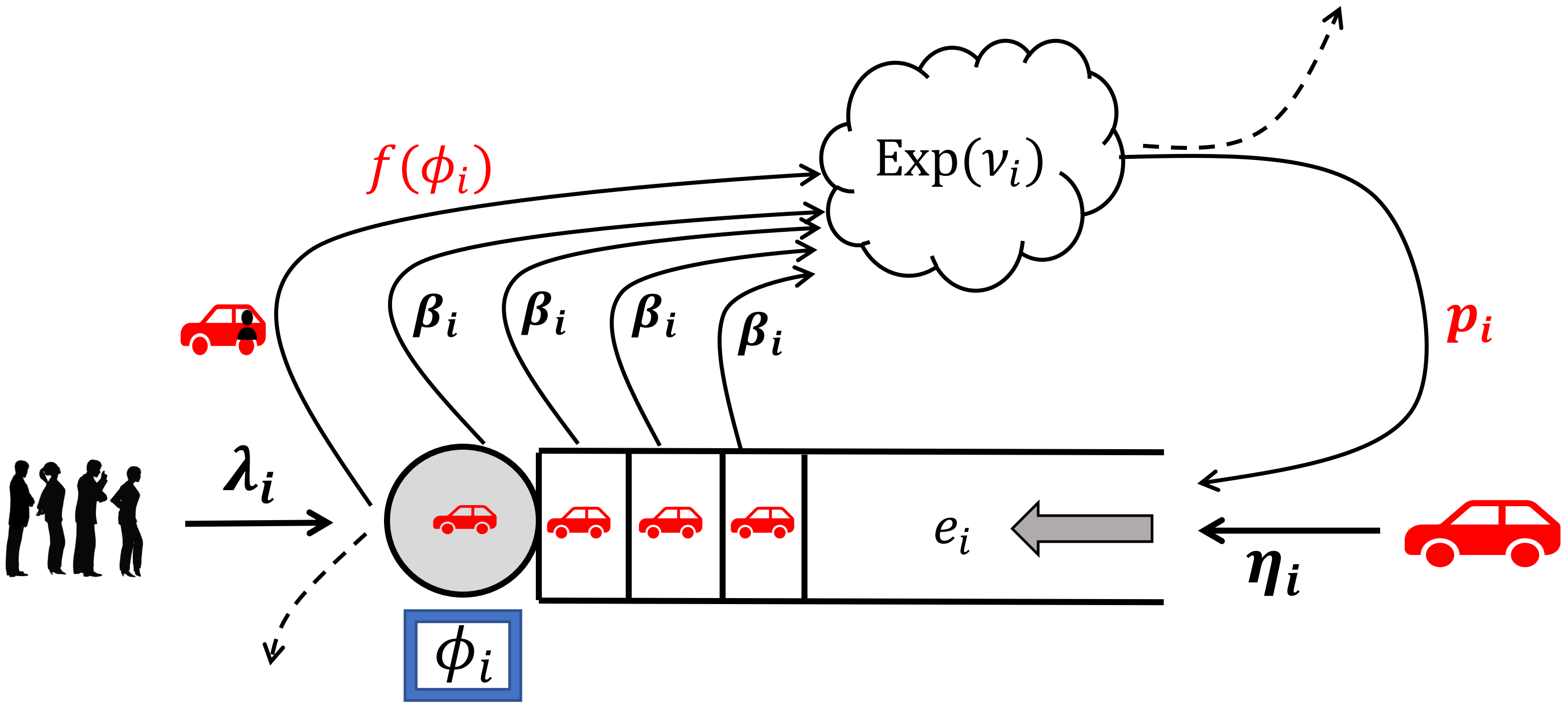


Passenger market share split based on QoS considerations

BCMP model for platform i



BCMP model for platform i



Can be cast as a **BCMP** network with 2 stations

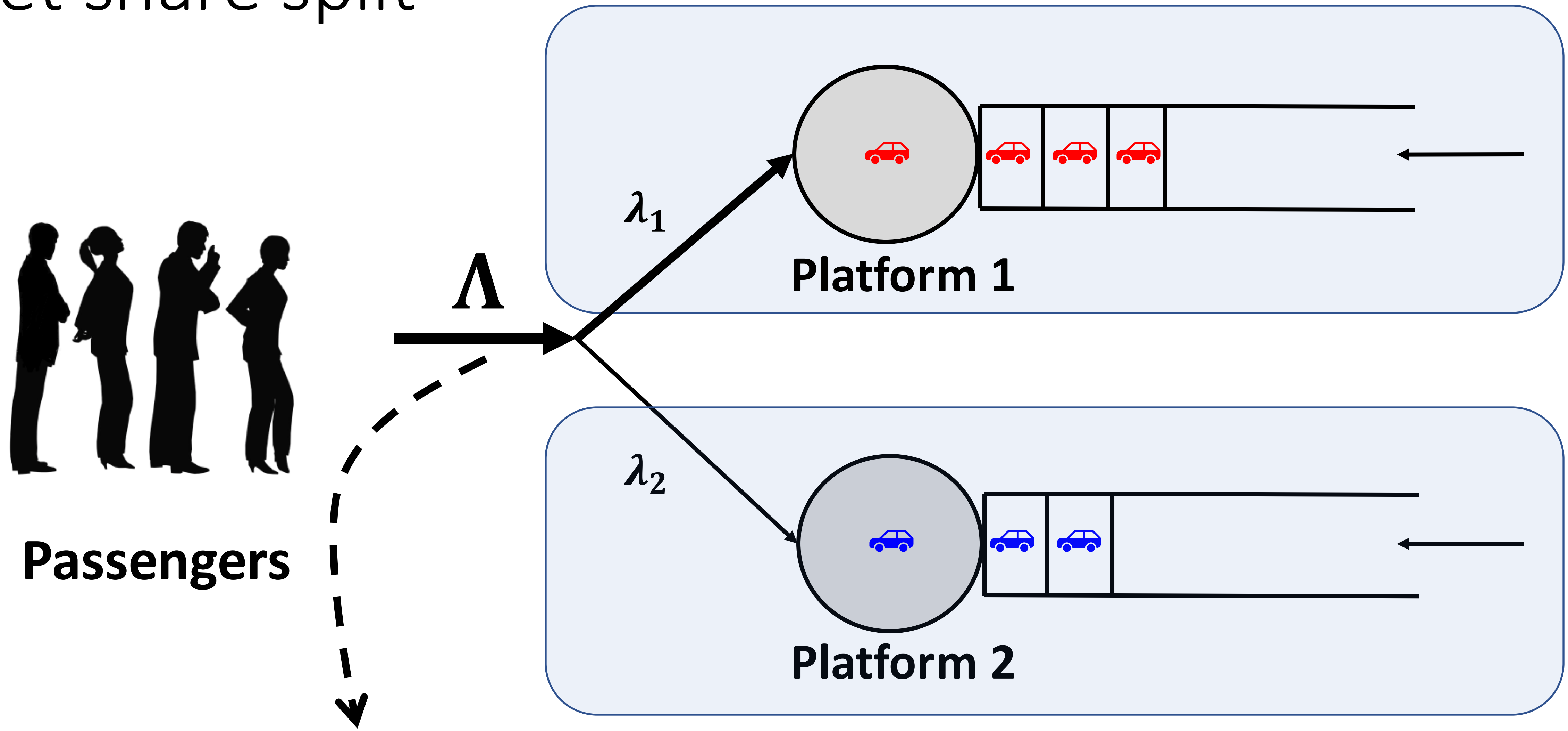
=> Product form stationary distribution

=> Formulation allows for:

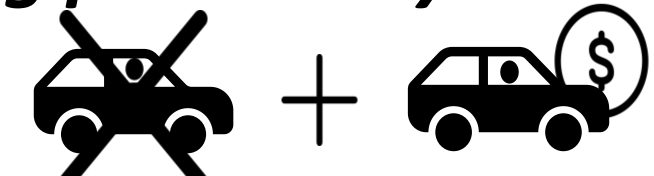
Dynamic (state dependent) pricing

Multiple zones (we consider only one)

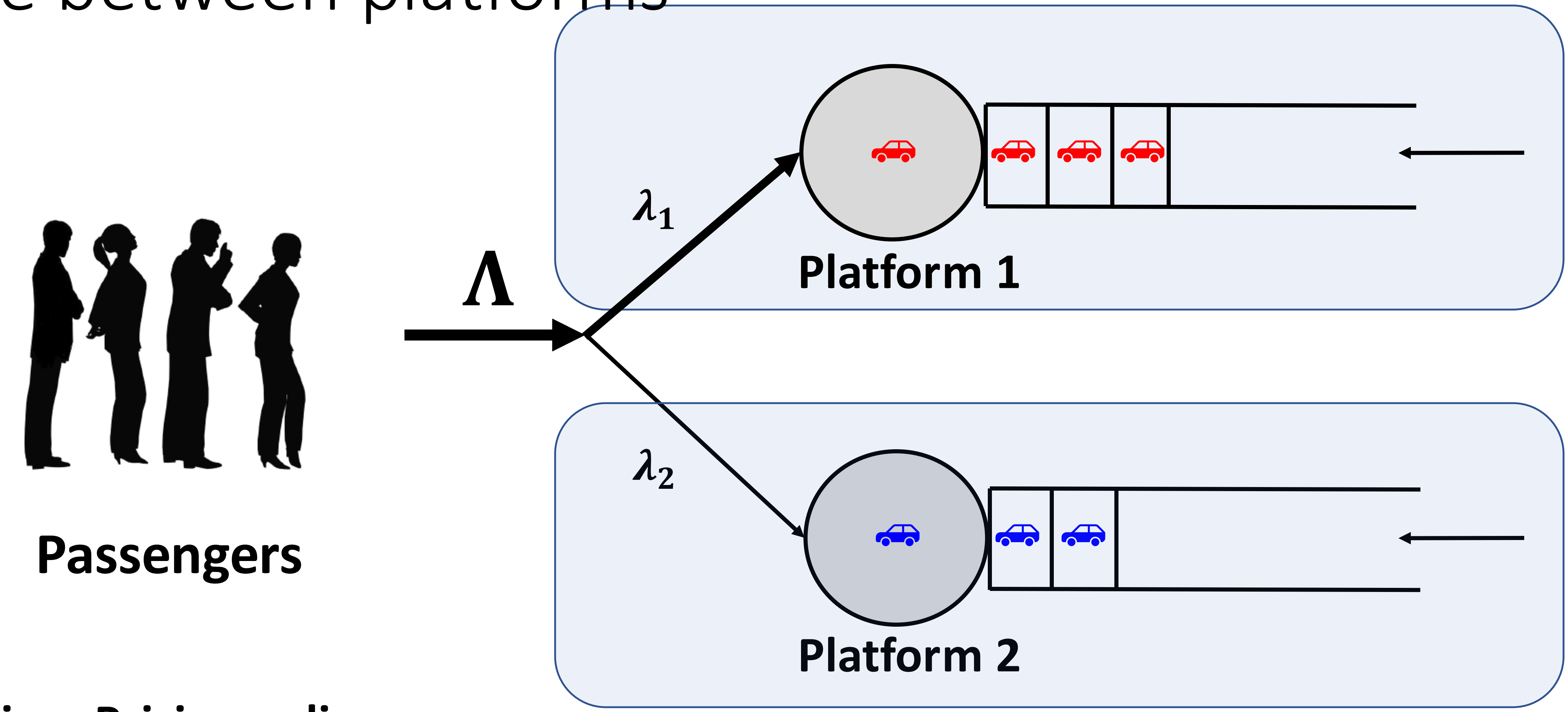
Market share split



Wardrop equilibrium wrt. ***overall blocking probability***



Game between platforms



Action: Pricing policy

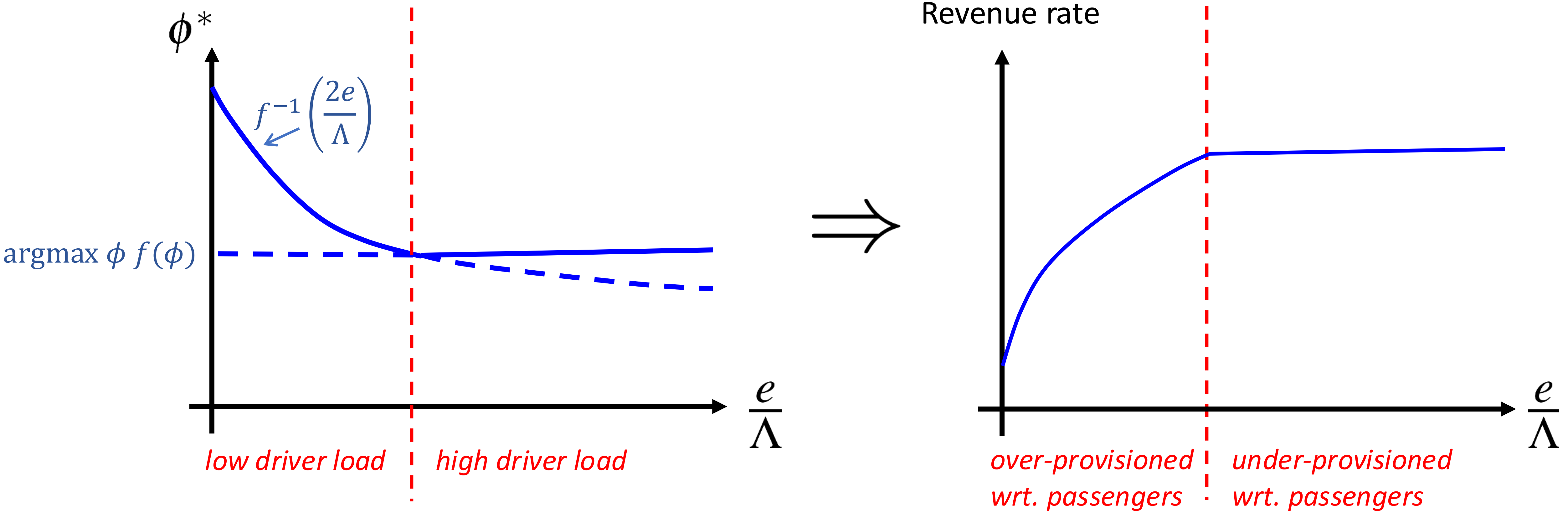
Payoff: Revenue rate

Platforms compete via market share segmentation

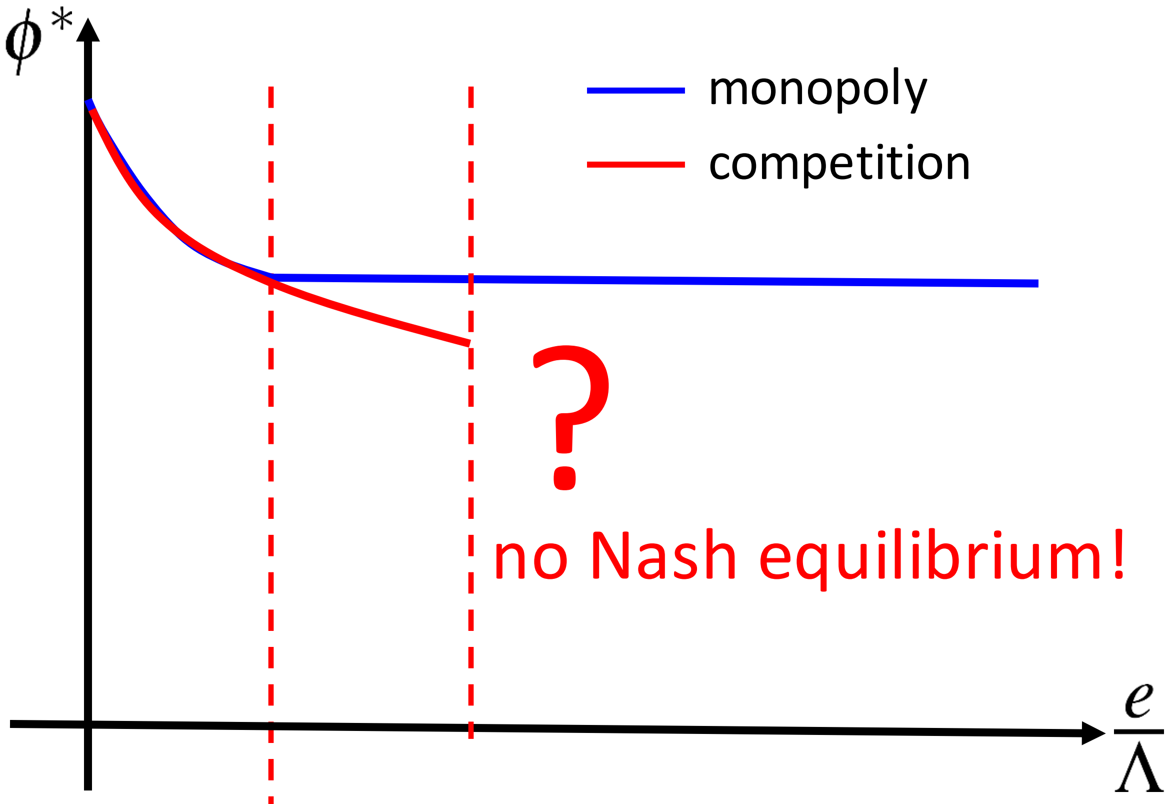
Simplifying assumptions

- Symmetric platforms
- Static pricing policy
- 'Limit system' where impatience $\beta \downarrow 0$

Monopoly ($\lambda_i = \Lambda/2$)

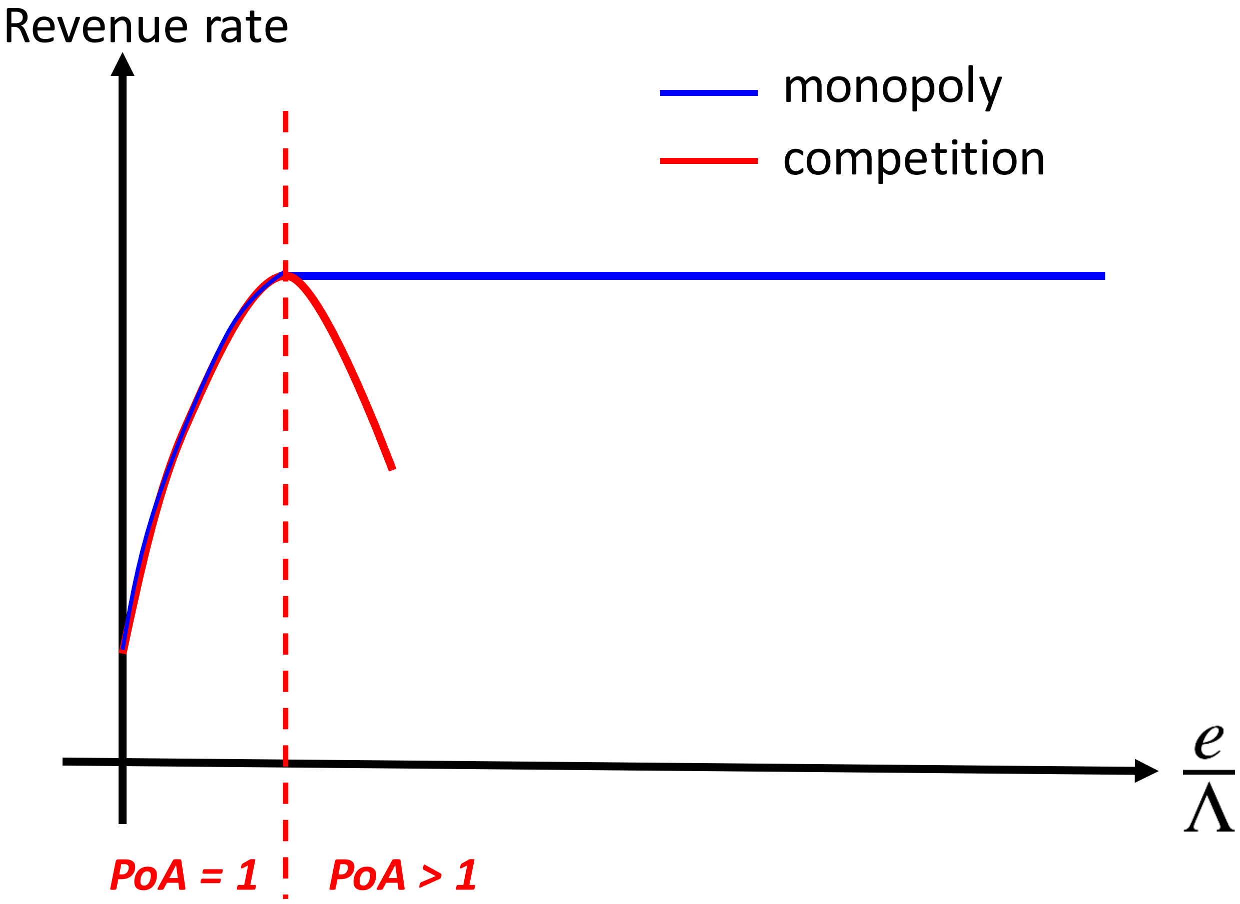


Competition between platforms

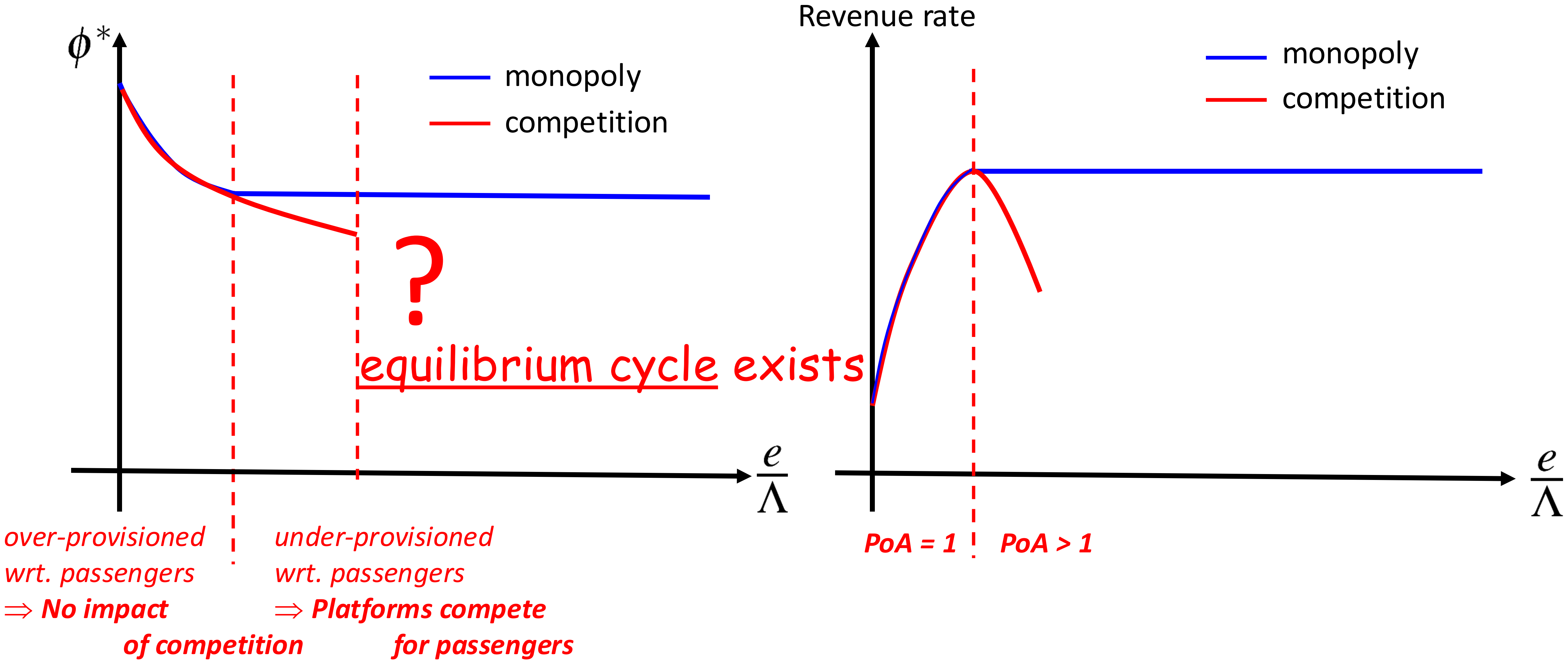


over-provisioned wrt. passengers
⇒ No impact of competition

under-provisioned wrt. passengers
⇒ Platforms compete for passengers



Competition between platforms



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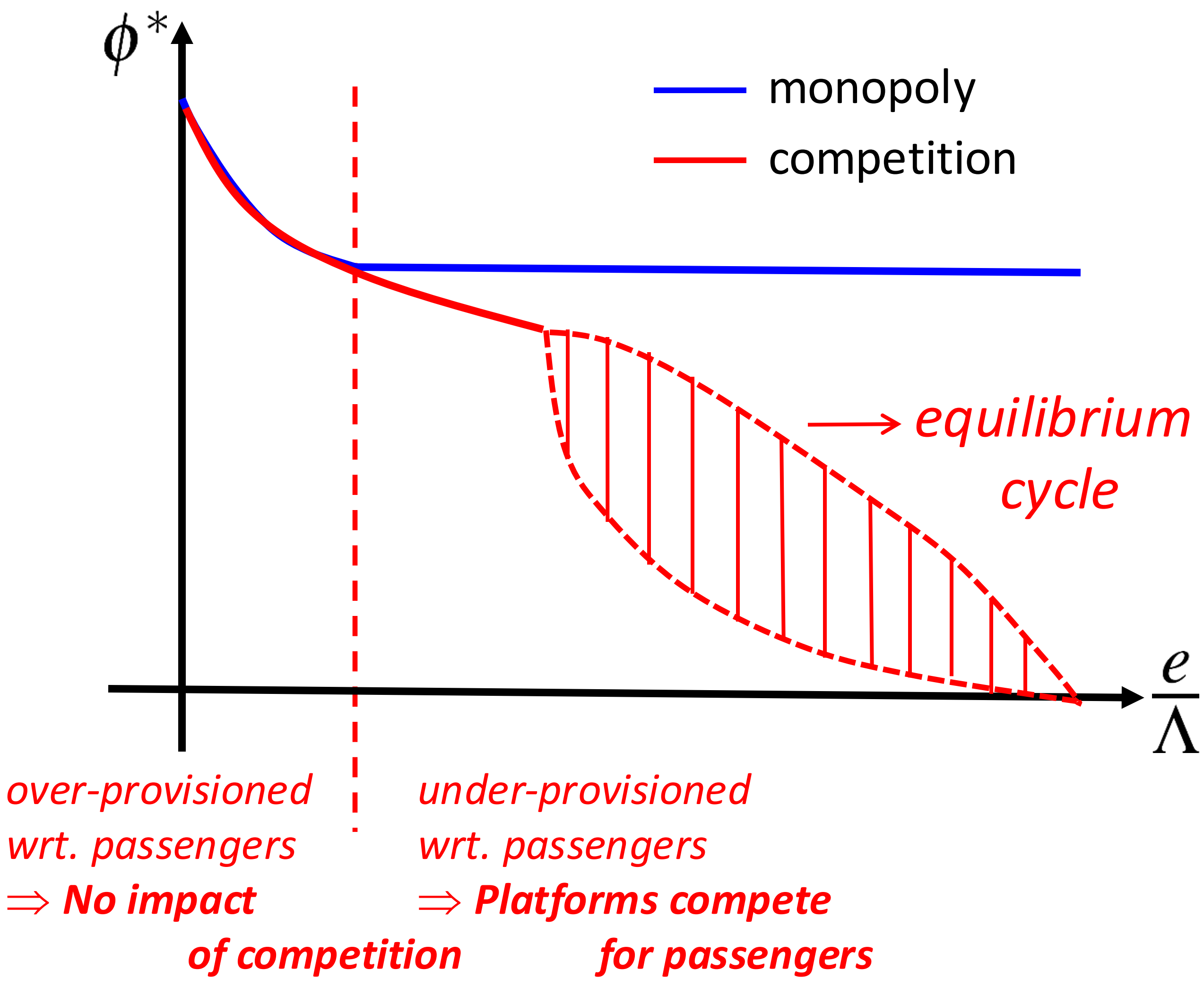
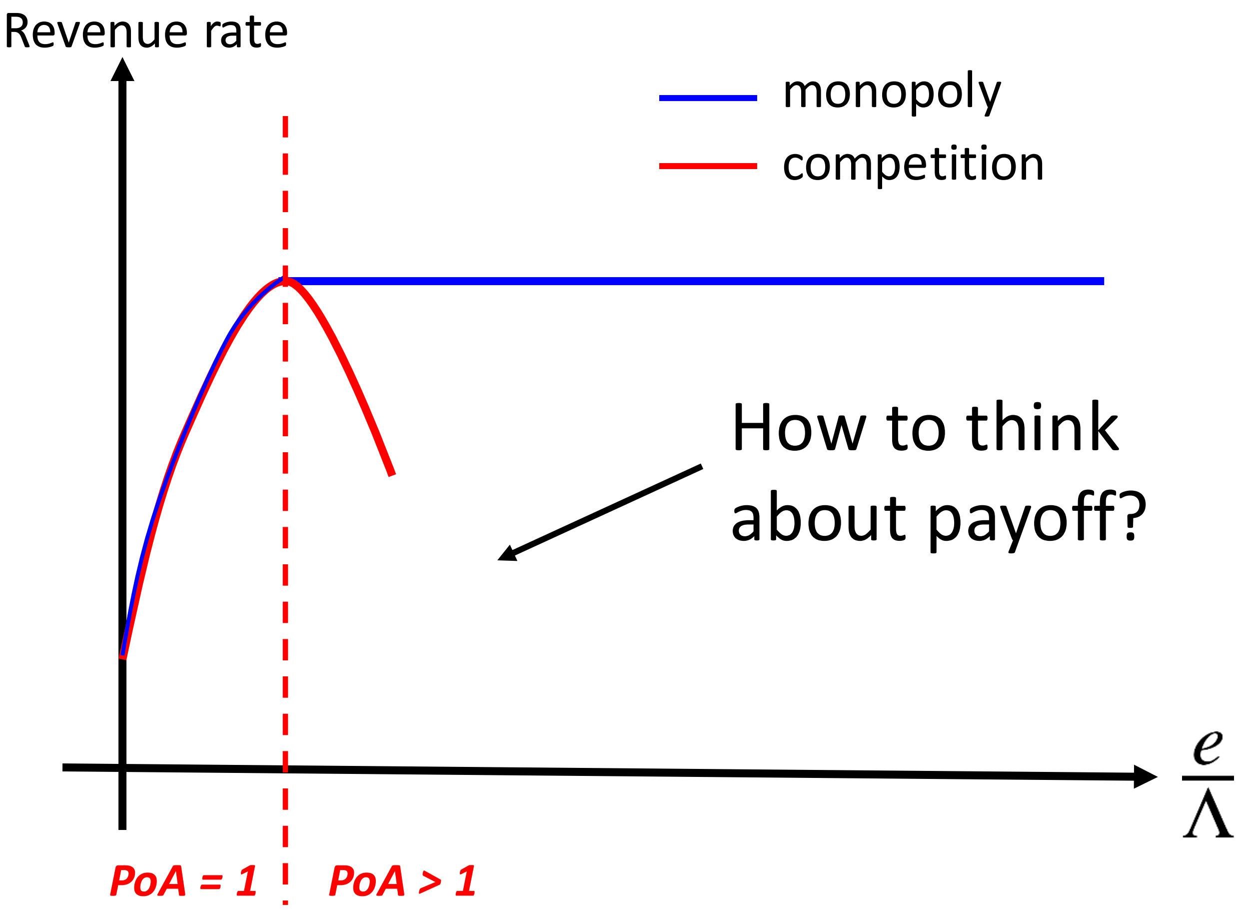




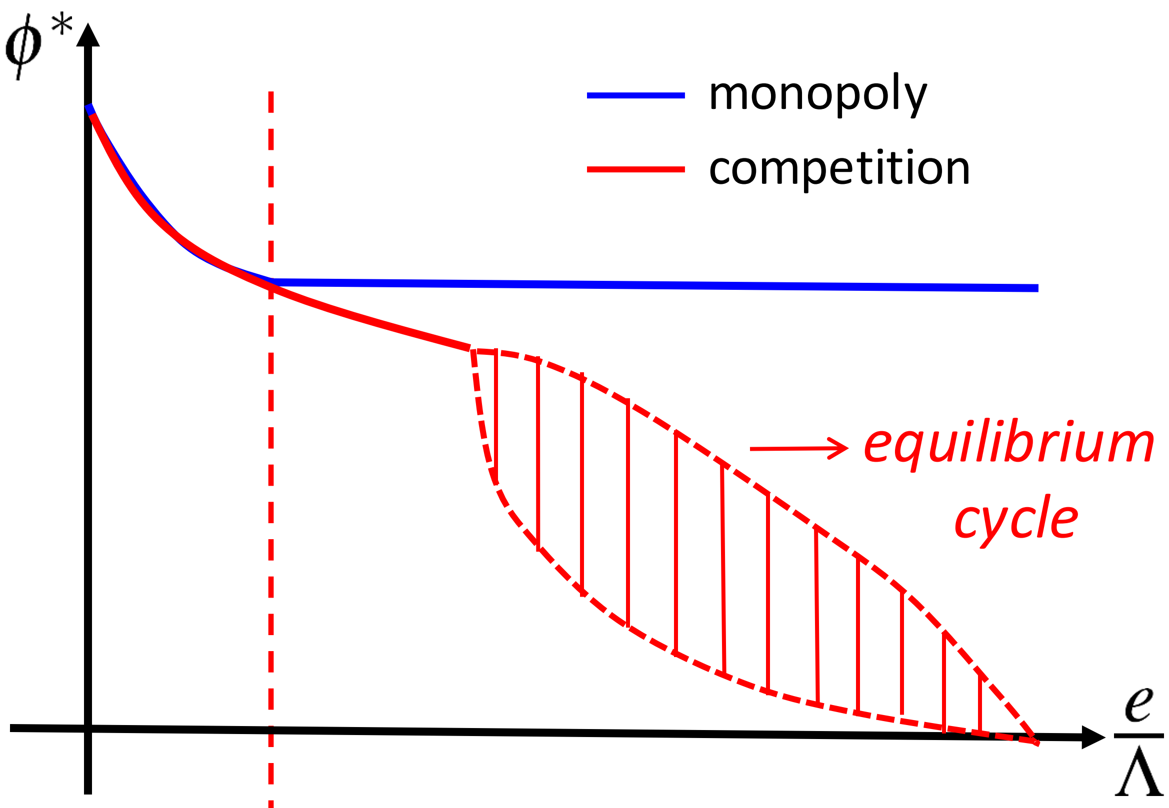
Security value for equilibrium cycle

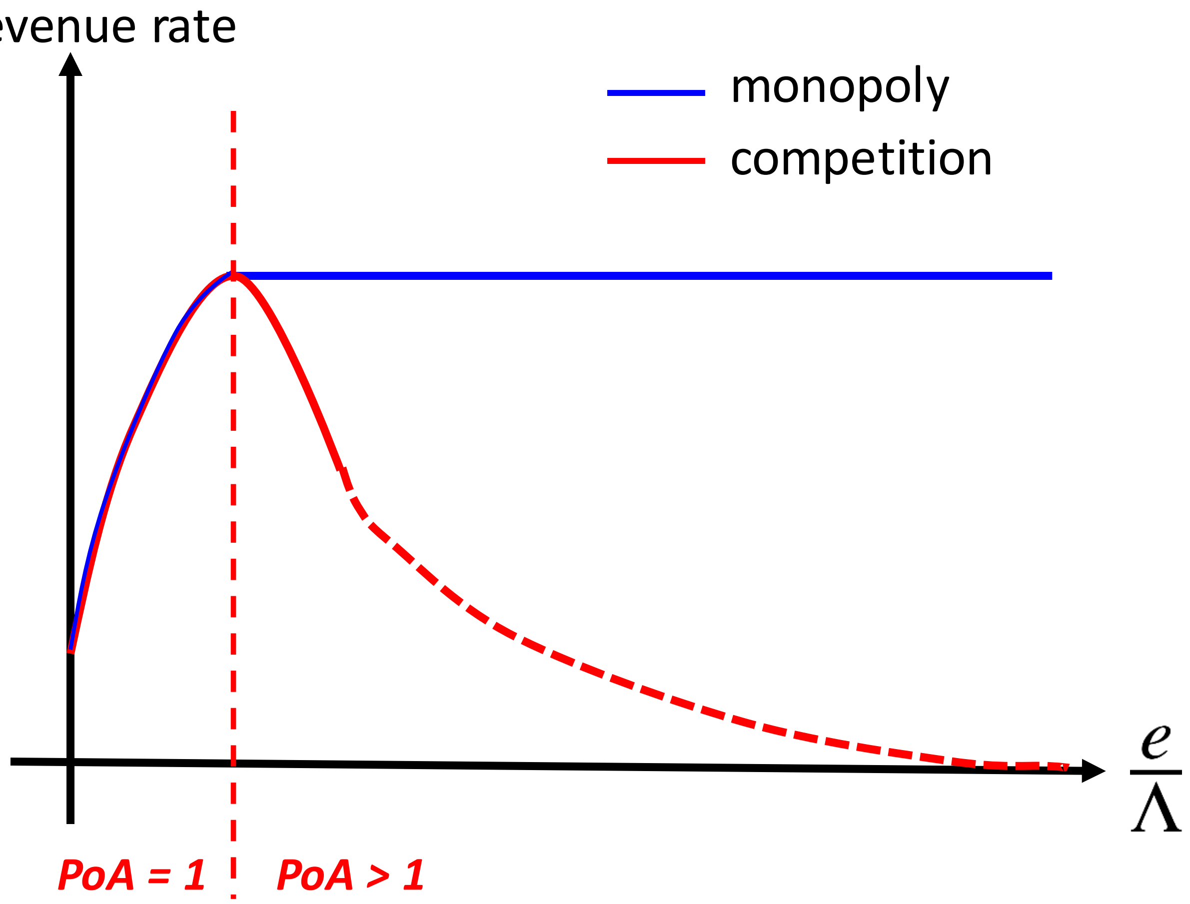
For player i, the *security value* corresponding to the equilibrium cycle is:

$$\underline{\mathcal{M}_i} = \max_{\phi_i \in \mathcal{E}_i} \min_{\phi_{-i} \in \mathcal{E}_{-i}} \mathcal{M}_i(\phi_i, \phi_{-i})$$

Interestingly, both platforms attain security value by playing either end-point of the equilibrium cycle!

Competition between platforms





over-provisioned wrt. passengers
⇒ ***No impact of competition***

under-provisioned wrt. passengers
⇒ ***Platforms compete for passengers***

PoA = 1 ***PoA > 1***

Summary

- Analyzed competition between ride hailing platforms
- When passengers are abundant, competition does not have an impact
- When passengers are scarce, platforms compete for them by lowering prices
- When passengers are really scarce, no Nash equililbrium.
  Equilibrium cycle instead!

Thank You