Observability of Discrete-Time LTI Systems Under Unknown Piece-Wise Constant Inputs

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***Abstract*—This letter is on observability of discrete-time LTI systems under unknown piece-wise constant inputs with sufficiently slow, but arbitrary update times. Assuming knowledge of the update times, we characterize the unobservable subspace and show that with sufficiently many measurements in each inter-update interval of the input, the unobservable subspace remains fixed. We explore the implications of the result for privacy in event-triggered control through an illustrative example.**

***Index Terms*—Networked control systems, sampled-data control, observability under unknown input, event-triggered control, privacy.**

I. Introduction

OBSERVABILITY under unknown inputs has been a topic of interest to the controls community for several decades. Motivated by the recent research trend of event-triggered control, we revisit the classical problem of observability.

A. Literature Review

The literature on observability of linear time invariant (LTI) systems under unknown or partially known inputs stretches back to late 1960s. Some early works on the topic are [1]–[4]. More recent works on the topic include sliding mode observer for unknown input and state estimation [5], observability under unknown inputs in the context of singular differential algebraic systems [6], structural input and state observability [7], time-delayed observers [8], [9] and in the context of switched systems [10], [11]. It is well known that if a continuous-time LTI system is observable under known inputs then periodic sampling retains that property except for some pathological sampling periods [12]. The increasing popularity of event-triggered control [13]–[16] raises the question of observability under aperiodic sampling, a topic on which there is currently very limited work [17].

The topic of this letter is also relevant for privacy in event-triggered control. While there exist some papers on privacy preserving or secure event-triggered control, such as [18]–[22], there is no work that studies the privacy implications of existing event-triggered controllers. Such a study is particularly important given that in event-triggered control there is implicit information in the event times about the state of the system [23].

Contributions: In this letter, we characterize the unobservable subspace of a discrete-time LTI system under unknown piece-wise constant input but known, possibly aperiodic, update times of the input. In particular, we give a result that if the updates in the input are slow enough then with sufficiently many measurements in each inter-update interval of the input, the unobservable subspace remains fixed with time. We apply this result to a system with event-triggered control in the presence of an eavesdropper (ED), which is an entity that seeks to determine private data, such as the state of the system. We assume that the ED has access to the sensor measurements, the control input update times and the triggering rule, which implicitly determines the input update times. We demonstrate, through an example, that if the triggering rule is event based then the state can be identified by ED up to a bounded set, whose "size" decreases with time to zero. This illustrates how such an ED can breach the privacy of the system under the knowledge of the event-triggering rule. On the other hand, for time-triggered updates of the input, this ED can infer nothing about the component of the state in the unobservable subspace.

In the context of observability under unknown inputs, to the best of our knowledge, there is no existing work on observability under an aperiodically updated, unknown piece-wise constant input. In the context of event-triggered control, this letter is the first one to explore the privacy implications of existing event-triggered controllers. Such a study is timely given the increasing popularity of event-triggered control.

Notation: We let $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}_0$ be the set of real numbers, integers and the set of natural numbers including zero, respectively. For $x \in \mathbb{R}^n$, we let $||x||$ be the Euclidean norm of x. We use the notation $[K_i, K_{i+1})_{\mathbb{Z}}$ for $[K_i, K_{i+1}) \cap \mathbb{Z}$. We use similar notations for closed, open and the other half-open intervals. We use $\mathbf{0}$ to denote zero matrices of appropriate dimensions. For vectors v and w, we use (v, w) to represent

Manuscript received August 31, 2020; revised November 13, 2020; accepted November 27, 2020. Date of publication December 15, 2020; date of current version March 1, 2021. This work was supported by Science and Engineering Research Board under Grant CRG/2019/005743. Recommended by Senior Editor J. Daafouz. *(Corresponding author: Pavankumar Tallapragada.)*

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Digital Object Identifier 10.1109/LCSYS.2020.3044587

the vector $[v^T,\ w^T]^T$. We use $M^\dagger$, $\mathrm{Ker}(M)$ and $\mathrm{Im}(M)$ for the pseudo-inverse, null-space and the image space of the matrix M, respectively. We let $\mathcal{Z}^\perp$ be the orthogonal complement of a subspace $\mathcal{Z}$. We use I_r to represent the identity matrix of dimension r.

II. Problem Formulation

We consider a discrete-time linear time-invariant system

$$x(k+1) = Ax(k) + Bu(k) \tag{1a}$$
$$y(k) = Cx(k) \tag{1b}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^p$ are the plant state, the control input and the measurement, respectively. We assume that the input is piece-wise constant, i.e.,

$$u(k) = u(K_i) \quad \forall k \in [K_i, K_{i+1})_{\mathbb{Z}}, \tag{1c}$$

where $\mathcal{K} := \{K_i\}_{i\in\mathbb{N}_0}$ is the increasing sequence of input *update times*. We call the set of time steps $[K_i, K_{i+1})_{\mathbb{Z}}$ as the i^{th} *inter-update interval*.

A. Assumptions

We make the following assumptions about the system (1) and the input (1c).

A1 Matrices B and C are full column rank and full row rank matrices, respectively.

A2 Pair (A, C) is observable.

A3 Input signal $u(.)$ is unknown. The update times K_i are known and the inter-update times satisfy $K_{i+1} - K_i \geq n+1$ for all $i \in \mathbb{N}_0$.

Note that there is no loss of generality in Assumption (A1), i.e., if B is not full column rank, then we can choose a matrix $\hat{B}$ whose columns form a basis for the column space of B and for each u there is a unique $\hat{u}$ such that $Bu = \hat{B}\hat{u}$. At the same time, for each $\hat{u}$ there is at least one u such that $Bu = \hat{B}\hat{u}$. Thus, the effect of the input u on the system is the same as that of $\hat{u}$. Similarly, assuming C is full row rank is only to ensure that there are no redundant outputs that are obtained as a linear combination of other outputs.

B. Reformulation as an Impulsive System

To study the unobservable subspace of the system (1) with unknown piece-wise constant input, as described in (1c), we reformulate the system into an autonomous system with state "jumps" at the input update times. We consider the unknown piece-wise constant input as an additional state variable $\hat{u}$. Hence, the augmented state of the system is $z(k) := (x(k), \hat{u}(k))$. During the i^{th} *inter-update interval*, $\hat{u}(k)$ is constant and it "jumps" to $u(k)$ at the time steps $k \in \{K_i\}$. Thus, letting

$$\bar{A} = \begin{bmatrix} A & B \\ \mathbf{0} & I_m \end{bmatrix}, \qquad \bar{C} = \begin{bmatrix} C & \mathbf{0} \end{bmatrix},$$

we can write the dynamics (1) as

$$z(k+1) = \begin{cases} \bar{A}z(k), & \forall (k+1) \notin \mathcal{K}, \\ \begin{bmatrix} Ax(k) + B\hat{u}(k) \\ u(k+1) \end{bmatrix}, & \forall (k+1) \in \mathcal{K} \end{cases} \tag{2a}$$
$$y(k) = \bar{C}z(k). \tag{2b}$$

Thus, the full system is given by the collection of A, B, C, $\mathcal{K}$ and $u(K)$ for all $K \in \mathcal{K}$.

Note that (2) is an exact reformulation of system (1). Thus, we can study the question of observability under unknown piece-wise constant $u(.)$, with known update times, in the context of the impulsive system (2). To systematically analyze this question, we introduce the following definition.

Definition 1: For the system (2), we define the unobservable subspace at time k given a horizon $w \geq k$, $\mathcal{Z}(w, k)$, as the set of all z such that there exist initial $x(0)$ and a piece-wise constant control input with input update times $\mathcal{K}$ such that $z(k) = z$ and the output is uniformly zero on $[0, w]_{\mathbb{Z}}$. Thus, formally $\mathcal{Z}(w, k)$ is the set

$$\begin{aligned} \{z \in \mathbb{R}^{n+m} : \exists x_0 \in \mathbb{R}^n,\ \exists u_{K_i} \in \mathbb{R}^m \quad \forall K_i \in \mathcal{K}. \\ \text{s.t. for } (2),\ x(0) = x_0,\ \hat{u}(K_i) = u_{K_i} \quad \forall K_i \in \mathcal{K}, \\ z(k) = z,\ y(j) = \mathbf{0} \quad \forall j \in [0, w]_{\mathbb{Z}}\}. \end{aligned} \tag{3}$$

C. Objectives

Under Assumptions (A1)-(A3), the objectives of this letter are the following.

1) Characterize the unobservable subspace and observability of system (1), equivalently (2), under unknown input.
2) Explore the implications for privacy in event-triggered networked control systems.

We explore this question in the following two sections. In Section III, we address the question under the assumption of constant but unknown input. Then, in Section IV we extend the analysis to the case of piece-wise constant unknown inputs but with known update times.

III. Observability Under Constant Unknown Input

Observability of system (2) with a constant unknown input can be studied with the observability matrix associated with the pair $(\bar{A}, \bar{C})$, i.e., $O(w)$, where

$$O(w) := \begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \bar{C}\bar{A}^2 \\ \bar{C}\bar{A}^3 \\ \vdots \\ \bar{C}\bar{A}^{w-1} \end{bmatrix} = \begin{bmatrix} C & \mathbf{0} \\ CA & CB \\ CA^2 & C(A+I)B \\ CA^3 & C(A^2+A+I)B \\ \vdots & \vdots \\ CA^{w-1} & C\sum_{i=0}^{w-2} A^i B \end{bmatrix}. \tag{4}$$

Clearly, observability of the system (1) under a constant unknown input, $\mathcal{K} = \{0\}$, is directly related to observability of system (2) in the classical sense. With $\mathcal{K} = \{0\}$, $\mathcal{Z}(w, k)$ in (3) reduces to

$$\begin{aligned} \mathcal{Z}(w, k) = \{z \in \mathbb{R}^{n+m} : \exists z_0 \in \mathbb{R}^{n+m} \text{ s.t. for } (2) \\ z(0) = z_0,\ z(k) = z,\ y(j) = \mathbf{0} \quad \forall j \in [0, w]_{\mathbb{Z}}\}. \end{aligned} \tag{5}$$

In particular, it is easy to see that under a constant unknown input, $\mathcal{Z}(w, 0) = \mathrm{Ker}(O(w))$. The following lemma characterizes $\mathrm{Ker}(O(w))$ and hence the unobservable subspace of the system (2) under constant unknown input.

Lemma 1: Suppose that Assumption (A2) holds. Then for all $w \geq n+1$, $\mathrm{Ker}(O(w)) = \mathrm{Ker}(R)$, where

$$R := \begin{bmatrix} (A - I_n) & B \\ C & \mathbf{0} \end{bmatrix}. \tag{6}$$

Thus, under a constant unknown input, $\mathcal{K} = \{0\}$,

$$\mathcal{Z}(w, 0) = \mathrm{Ker}(O(w)) = \mathrm{Ker}(R) \quad \forall w \geq n+1.$$

Proof: From (4), we see that $z = (x, \hat{u}) \in \mathrm{Ker}(O(w))$ iff

$$CA^i x = \mathbf{0}, \text{ for } i = 0, \tag{7a}$$

$$CA^i x + C\left(\sum_{j=0}^{i-1} A^j\right) B\hat{u} = \mathbf{0} \quad \forall i \in [1, w-1]_{\mathbb{Z}}. \tag{7b}$$

Subtracting the i^{th} from the $(i+1)^{\text{th}}$ equation in (7), we get

$$CA^i((A - I_n)x + B\hat{u}) = \mathbf{0} \quad \forall i \in [0, w-2]_{\mathbb{Z}}. \tag{8}$$

Under Assumption (A2), the only vector v that satisfies $CA^i v = \mathbf{0}$ for all $i \in [0, n-1]_{\mathbb{Z}}$ is $v = \mathbf{0}$. Thus, (8) and (7a) imply that $\mathrm{Ker}(O(w)) = \mathrm{Ker}(R)$ $\forall w \geq n+1$. The result now follows as $\mathcal{Z}(w, 0) = \mathrm{Ker}(O(w))$ if $\mathcal{K} = \{0\}$. ■

Remark 1: Notice that the matrix R in (6) is in essence the Rosenbrock's system matrix [24] of system (1) for the discrete-time DC frequency. This is not surprising since in Lemma 1, we seek precisely the "transmission blocking" plant states and constant control inputs.

Also, given that $O(w)$ has $n+m$ columns, one may expect that $\mathcal{Z}(w, 0)$ would, in general, remain constant only for $w \geq n+m$. However, Lemma 1 in fact says that $\mathcal{Z}(w, 0)$ remains constant for all $w \geq n+1$.

In the following result, we give a simple property of $\mathrm{Ker}(O(w))$ that plays a very important role in the setting of piece-wise constant unknown input.

Corollary 1: Suppose that Assumptions (A1) and (A2) hold and $w \geq n+1$. Then the following statements are true:

(a) If $(x, u_1) \in \mathrm{Ker}(O(w))$ and $(x, u_2) \in \mathrm{Ker}(O(w))$ then $u_1 = u_2$.

(b) If $(x_1, u) \in \mathrm{Ker}(O(w))$ and $(x_2, u) \in \mathrm{Ker}(O(w))$ then $x_1 = x_2$.

Proof: From Lemma 1, we know that $\mathrm{Ker}(O(w)) = \mathrm{Ker}(R)$. Then, claim (a) follows from the full column rank of B in Assumption (A1). Claim (b) follows from observabiliy of the pair (A, C) in Assumption (A2), which implies that $\begin{bmatrix} (A - I_n) \\ C \end{bmatrix}$ has full column rank. ■

We now go on to characterize $\mathcal{Z}(w, k)$ when $\mathcal{K} = \{0\}$. In particular, if $\mathcal{K} = \{0\}$, (5) indicates that the dimension of $\mathcal{Z}(w, k)$ is no larger than that of $\mathcal{Z}(w, 0)$. Further, we also know that $\mathcal{Z}(w, 0)$ is an $\bar{A}$-invariant subspace [24] for all $w \geq n+1$. Thus, we can say that $\mathcal{Z}(w, k) \subseteq \mathcal{Z}(w, 0)$. But the following result says that $\mathcal{Z}(w, k) = \mathcal{Z}(n+1, 0)$ for all $k \geq 0$ and $w \geq n+1$.

Theorem 1: Consider the system (2) with a constant unknown input and suppose Assumptions (A1) and (A2) hold. Further, suppose that $z(0) \in \mathrm{Ker}(R)$. Then $z(k) = z(0)$ for all $k \geq 0$. As a consequence, $\mathcal{Z}(w, k) = \mathcal{Z}(n+1, 0)$ for all $k \geq 0$ and $w \geq n+1$.

Proof: Since $\mathcal{Z}(w, 0)$ is an $\bar{A}$-invariant subspace, we know that $\mathcal{Z}(w, k) \subseteq \mathcal{Z}(w, 0)$, and from Lemma 1, we know that $\mathcal{Z}(w, 0) = \mathcal{Z}(n+1, 0) = \mathrm{Ker}(R)$ for all $w \geq n+1$. Thus, it suffices to show that $\mathrm{Ker}(R) \subseteq \mathcal{Z}(w, k)$.

Now, since $z(0) = (x(0), \hat{u}(0)) \in \mathrm{Ker}(R)$, we have

$$x(1) = Ax(0) + B\hat{u}(0) = x(0).$$

Further, as the control input is constant, $\hat{u}(k) = \hat{u}(0)$ for all $k \geq 0$. Further, if $x(k) = x(0)$, we have

$$x(k+1) = Ax(k) + B\hat{u}(k) = Ax(0) + B\hat{u}(0) = x(0). \tag{9}$$

Using mathematical induction, we conclude that $z(k) = z(0)$ $\forall k \in \mathbb{N}_0$. Thus, (5) implies that $z(0) \in \mathcal{Z}(w, k)$, that is $\mathcal{Z}(w, 0) = \mathrm{Ker}(R) \subseteq \mathcal{Z}(w, k)$, which then means that $\mathcal{Z}(w, k) = \mathcal{Z}(w, 0)$ for all $k \geq 0$ and $w \geq n+1$. ■

Note that, this result holds even if A is singular. Further, it is interesting that if $z(0) \in \mathcal{Z}(w, 0)$ with $w \geq n+1$, then $z(k) = z(0)$ for all $k \geq 0$, which goes beyond $\bar{A}$-invariance of the set $\mathcal{Z}(w, 0)$. As we will see, this has an interesting implication for observability under an unknown piece-wise constant input, which is our next topic of discussion.

IV. Observability Under Piece-Wise Constant Unknown Input

We now seek to characterize $\mathcal{Z}(w, k)$ under an unknown piece-wise constant input, as given in (3). Given the extra degrees of freedom provided by u_{K_i} for $K_i \in \mathcal{K}$ it seems plausible, unlike in the constant input case, that in general $\mathcal{Z}(w, k)$ may not be a subset of $\mathcal{Z}(w, 0)$.

Remark 2: Due to the causal nature of the system (2), we can say that, for all $w \in \mathbb{N}_0$ and for all $k \leq w$, $\mathcal{Z}(w, k)$ does not depend on update times greater than w. Thus, we define the truncated set of *update times up to w* as

$$\mathcal{K}_w := \{K \in \mathcal{K} : K \leq w\} \cup \{w\} =: \{K_0, K_1, \ldots, K_{N(w)}\},$$

which is the set of all update times up to and including w. Note that we include $K_{N(w)} = w \in \mathcal{K}_w$ even if $w \notin \mathcal{K}$. However, as the input $u(w)$ can only affect the outputs $y(k)$ for $k > w$, we see that $\mathcal{Z}(w, k)$ is unaffected by whether $w \in \mathcal{K}$ or not. Hence, we can obtain $\mathcal{Z}(w, k)$ by supposing $\mathcal{K} = \mathcal{K}_w$ has only finitely many input updates.

Now, in order to characterize $\mathcal{Z}(w, k)$, let

$$\mathbf{y}^i := (y(K_i),\ y(K_i + 1),\ \ldots,\ y(K_{i+1} - 1)),$$

which is the vector containing all the measurements in the i^{th} inter-update interval in $\mathcal{K}_w$. Then, we can write

$$\mathbf{y}^i = O(q_i)z(K_i),\ q_i := K_{i+1} - K_i \quad \forall i \in [0, N(w))_{\mathbb{Z}}. \tag{10}$$

Although $z(K_i) = (x(K_i), \hat{u}(K_i))$, note that in (10), only $x(0)$ and $\hat{u}(K_i)$ can be chosen arbitrarily. The rest of $x(K_i)$ are determined by the dynamics (2). In particular, from variation of constants, we know that

$$x(K_{i+1}) = A^{K_{i+1}-K_i}x(K_i) + \sum_{j=0}^{K_{i+1}-K_i-1} (A^j B)\hat{u}(K_i). \tag{11}$$

Now, we can say that

$$\begin{aligned}\mathcal{Z}(w,k) = \{z \in \mathbb{R}^{n+m} : \exists v_i \in \mathbb{R}^{(n+m)},\ \forall i \in [0, N(w))_{\mathbb{Z}},\\ \text{s.t. for } (2),\ z(K_i) = v_i,\ (11),\ z(k) = z,\\ Cx(w) = \mathbf{0},\ O(q_i)z(K_i) = \mathbf{0} \quad \forall i \in [0, N(w))_{\mathbb{Z}}\}. \quad (12)\end{aligned}$$

Now, we are ready to present our results on observability under unknown piece-wise constant control.

Theorem 2: Consider system (2) with unknown piece-wise constant input and suppose Assumptions (A1)-(A3) hold. Further, let $w \in \mathbb{N}_0$ be such that $w - K_i \geq n + 1$ for all $K_i \in \mathcal{K}_w \setminus \{w\}$. Finally, suppose that $k \in [0, w]_{\mathbb{Z}}$. If $z(k) = z = (x, \hat{u}) \in \mathcal{Z}(w, k)$ then $z(j) = z = (x, \hat{u})$ for all $j \in [0, w - 1]_{\mathbb{Z}}$ and $x(w) = x$. As a consequence, $\mathcal{Z}(w, k) = \text{Ker}(O(n + 1)) = \text{Ker}(R)$ for all $k \in [0, w]_{\mathbb{Z}}$.

Proof: Our starting point is (12). Observe that Assumption (A3), Lemma 1 and the fact that $w - K_i \geq n + 1$ for all $K_i \in \mathcal{K}_w \setminus \{w\}$ together imply that $O(q_i)z(K_i) = \mathbf{0}$ iff $z(K_i) \in \text{Ker}(O(n + 1)) = \text{Ker}(R)$ for all $i \in [0, N(w))_{\mathbb{Z}}$. This fact together with (11) and the induction similar to the one in (9) implies that $x(K_{i+1}) = x(K_i)\ \forall i \in [0, N(w))_{\mathbb{Z}}$. Further, Corollary 1(a) implies that $\hat{u}(K_{i+1}) = \hat{u}(K_i)$ and hence $z(K_i) = z(K_0)$ for all $i \in [0, N(w))_{\mathbb{Z}}$. Now, applying Theorem 1 on each of the inter-update intervals in $\mathcal{K}_w$ in isolation and using (12), we obtain the first claim of the result. The second claim is now a consequence of (12) and the fact that $z(k) = z(K_0) \in \text{Ker}(O(n + 1))$. ∎

Note that Theorem 2 allows the possibility that $\mathcal{Z}(w, k)$ can be something other than $\text{Ker}(O(n + 1))$ for w that violate the assumption that $w - K_i \geq n + 1$ for all $K_i \in \mathcal{K}_w \setminus \{w\}$. For all other w, Theorem 2 says that the unobservable subspace is the same. Given this, we let

$$\mathcal{Z} := \text{Ker}(O(n + 1)) \ = \text{Ker}(R).$$

Further, for brevity, we also let $O{:=}O(n + 1)$.

Next, we want to define the *known* and the *unknown* parts of the state and the control input. To this end, letting

$$y(k : k + j) := (y(k), \ldots, y(k + j)),$$

we can write the output relation as

$$y(K_i : K_i + n) = Oz(K_i) =: O_1x(K_i) + O_2\hat{u}(K_i) \quad \forall K_i \in \mathcal{K},$$

where O_1 and O_2 are the first n and last m columns of the matrix O, respectively, such that $O =: [O_1 \ \ O_2]$. Note that by row operations, the last n block rows of O_2 can be reduced to the first n block rows of O_1B. Then, Assumptions (A1)- (A2) imply that O_1 and O_2 have full column rank. Hence, there exists a unique $\hat{u}(K_i)$ compatible with each pair of $x(K_i)$, and a feasible output sequence $y(K_i : K_i + n)$. Further,

$$\hat{u}(K_i) = O_2^{\dagger}\big[y(K_i : K_i + n) - O_1x(K_i)\big].$$

Definition 2: We denote the *known* and the *unknown* parts of $z(k)$ with $r(k)$ and $\zeta(k)$, respectively, which we define as

$$\begin{aligned}&r(0) := O^{\dagger}y(0 : n),\ \zeta(0) \in \mathcal{Z},\ \text{ s.t. } z(0) = r(0) + \zeta(0)\\ &(x^{\zeta}(k), \hat{u}^{\zeta}(k)) := \zeta(k) := \bar{A}^k\zeta(0) \quad \forall k \in \mathbb{N}_0\\ &(x^r(k), \hat{u}^r(k)) := r(k) := \bar{A}^{k-K_i}r(K_i) \quad \forall k \in [K_i, K_{i+1})_{\mathbb{Z}},\end{aligned}$$

where

$$\begin{aligned}x^r(K_i) &:= \big[I_n \ \ \mathbf{0}\big]\bar{A}^{(K_i-K_{i-1})}r(K_{i-1})\\ \hat{u}^r(K_i) &:= O_2^{\dagger}\big[y(K_i : K_i + n) - O_1x^r(K_i)\big].\end{aligned}$$

We also call $x^{\zeta}(k)$, $\hat{u}^{\zeta}(k)$ as the unknown and $x^r(k)$, $\hat{u}^r(k)$ as the known parts in plant states and input, respectively. •

Note that $r(0)$ can only be computed after $n + 1$ measurements. Similarly, for each $K_i \in \mathcal{K}$, $\hat{u}^r(K_i)$ depends on $y(K_i : K_i + n)$. This implies that $r(k)$ can only be evaluated with an initial delay of $n + 1$ time-steps in each inter-update interval, that is to say that $r(k)$ can only be evaluated at time step $\max\{k, L_k + n\}$, where $L_k := \max\{K \in \mathcal{K} : K < k\}$. On the other hand, $\zeta(k)$ cannot be determined only from the measurements though we know that $\zeta(k)$ remains $\zeta(0)$ for all k. In the next result, we show that the known and the unknown parts, $r(k)$ and $\zeta(k)$, add up to $z(k)$ for all $k \geq 0$.

Corollary 2: Consider the system (2) under piece-wise constant unknown input and suppose Assumptions (A1)-(A3) hold. Then $\zeta(k) = \zeta(0)$ and $z(k) = r(k) + \zeta(k)\ \forall k \in \mathbb{N}_0$.

Proof: Theorem 1 ensures that $\zeta(k) = \zeta(0)\ \forall k \geq 0$. Next, we show by induction that $z(K_i) = r(K_i) + \zeta(K_i)\ \forall K_i \in \mathcal{K}$, which along with (2) implies that $z(k) = r(k) + \zeta(k)$ for all $k \in [K_i, K_{i+1})_{\mathbb{Z}}\ \forall K_i \in \mathcal{K}$. By definition $r(0) \in \mathcal{Z}^{\perp}$ and hence $z(0) = r(0)+\zeta(0)$. Now suppose that $z(K_i) = r(K_i)+\zeta(K_i)$ for $K_i \in \mathcal{K}$. Then the definition of $x^r(K_{i+1})$ implies that $x(K_{i+1}) - x^r(K_{i+1}) = x^{\zeta}(K_{i+1}) = x^{\zeta}(0)$. Next, since

$$Oz(K_{i+1}) = y(K_{i+1} : K_{i+1} + n) = Or(K_{i+1}),$$

we can say that $(z(K_{i+1}) - r(K_{i+1})) \in \mathcal{Z}$. Then, Corollary 1(a) implies that $\hat{u}(K_{i+1}) - \hat{u}^r(K_{i+1}) = \hat{u}^{\zeta}(K_{i+1}) = \hat{u}^{\zeta}(0)$ as $(x^{\zeta}(0), \hat{u}^{\zeta}(0)) \in \mathcal{Z}$. ∎

The known part $r(k)$ can be thought of as the estimate of the plant state and the unknown input given sufficient measurements. Theorem 2 and Corollary 2 indicate that the uncertainty about the unknown part $\zeta(k)$ cannot be reduced, from the subspace $\mathcal{Z}$, after the first $n+1$ time steps even if there are many updates to the control. However, with additional information such as the triggering rule in event-triggered control, we show in the next section that this uncertainty can be reduced. As a result, there can be a loss of privacy in event-triggered control. In contrast, in time-triggered control, there is no additional information in the update times and hence the uncertainty remains a subspace.

V. Privacy Implications for Event-Triggered Stabilization

In this section, we explore the implications of the results in Section IV for privacy in event-triggered stabilization. Through an example, we show that uncertainty about the unknown part can be reduced to a bounded subset of $\mathcal{Z}$ in finite time.

We consider a system with event-triggered state feedback transmissions from the plant to the controller over a network and in the presence of an eavesdropper (ED). We depict this setup in Figure 1. We assume that the eavesdropper has knowledge about the system matrices A, B and C in (1) and the event-triggering rule (ET). While these can be known even offline, we also assume that ED has access to some online

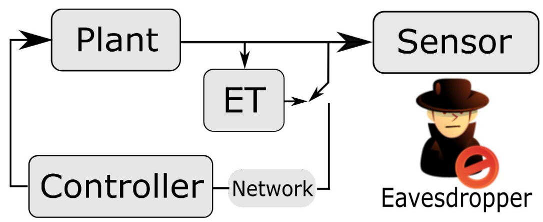

Fig. 1. Event-triggered control in the presence of an eavesdropper.

information, namely the sensor measurements and the event times $\{K_i\}$. However, we assume that ED cannot measure the plant state $x(.)$ or the control input $\hat{u}(.)$.

Consider system (1) with the pair (A, B) stabilizable and a matrix S such that $(A + BS)$ is schur stable. We let the input be a zero-order hold control

$$\hat{u}(k) = Sx(K_i) \quad \forall k \in [K_i, K_{i+1})_{\mathbb{Z}}, \tag{13}$$

where $\mathcal{K} := \{K_i\}_{i \in \mathbb{N}_0}$ is the increasing sequence of input update times determined implicitly by an event-triggering rule. We assume that ED has knowledge of the update times K_i when they occur. However, we assume that ED has no knowledge about the matrix S or even the form of the control, except that it is piece-wise constant.

We consider the triggering rule from [25], where an event occurs at time step k, i.e., $K_{i+1} = k$, if

$$||x(K_i) - x(k)|| \geq \mu ||x(k)||. \tag{14}$$

Reference [25] provides a range of values of μ for which the triggering rule (14) ensures asymptotic stabilization of the plant state to the origin. We define $\zeta = (x^{\zeta}, \hat{u}^{\zeta}) := \zeta(0)$ for brevity. Then, from Corollary 2, $x(k) = x^r(k) + x^{\zeta}$. Thus, the event-triggering rule can be written as: $K_{i+1} = k$ if

$$||x^r(k) + x^{\zeta}|| \leq \frac{1}{\mu} ||x^r(K_i) - x^r(k)||. \tag{15}$$

We assume that system (1) and the update times $\mathcal{K}$ generated by the triggering rule (14) satisfy Assumptions (A1)-(A3). Assumption (A3) is not restrictive in this context as one could choose a small enough sampling period for time-discretizing the underlying continuous time system in order to ensure Assumption (A3) is satisfied. With this review of event-triggering rule, we now look at the privacy implications for this stabilization task.

A. Privacy Implications

We consider plant state to be confidential information and hence a matter of privacy. Specifically, the smaller the error bound on ED's estimate of the plant state the greater is the loss of privacy. We assume that ED can accurately evaluate $x^r(k)$, the known part of the plant state $x(k)$. Hence the uncertainty in ED's estimation of the states is entirely due to the unknown part in the state. We assume that ED has access to the information $\mathcal{I}(k)$ at time k, where

$$\mathcal{I}(k) := \{A, B, C, \{y(j)\}_{j=0}^{k}, \mathcal{K}_k, \text{ ET rule (14)}\}.$$

We let $\mathcal{L}(k)$ be the uncertainty set at time-step k, which is the set of all possible values of x^{ζ} that are compatible with information available to ED up to time k. Then, we measure the breach in privacy through the "size" of these uncertainty sets as a function of time step k.

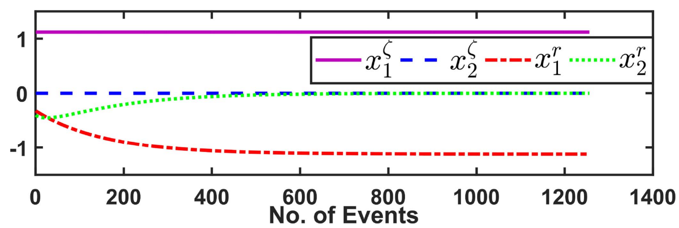

Fig. 2. Evolution of the known and the unknown parts of the plant state at the event times in the event-triggered stabilization task. Here x_i^{ζ} and x_i^r denote the i^{th} component in the vectors $x^{\zeta}(K_i)$ and $x^r(K_i)$. We see that the unknown part in plant states remains time invariant and the known part in the plant states evolves such that $\lim_{K_i \to +\infty} x^r(K_i) + x^{\zeta} = 0$.

Notice that sensor output measurements alone cannot reduce the uncertainty set $\mathcal{L}(k)$ to something smaller than $\mathcal{X}$, which is the projection of $\mathcal{Z}$ onto the plant-state space. Thus, there is a reduction only at the event times K_i. Hence, we consider $\mathcal{L}(k)$ only for $k = K_i \in \mathcal{K}$. In particular, using (15), which is equivalent to the ET rule (14), we first define $\mathcal{S}(K_i)$ as the set of all x^{ζ} compatible with (15) at $k = K_i$. Thus,

$$\mathcal{S}(K_i) := \{x \in \mathcal{X} : ||x + x^r(K_i)|| \leq b(i)\},$$

where $b(i) := \frac{1}{\mu} ||x^r(K_{i-1}) - x^r(K_i)||$. Then,

$$\mathcal{L}(K_i) := \cap_{j=1}^{i} \mathcal{S}(K_j), \tag{16}$$

and non-increasing with events. If in the event-triggering rule (14) or equivalently (15) μ is such that it ensures asymptotic stability of the origin of the plant state x then the uncertainty sets $\mathcal{L}(K_i)$ are bounded, $\lim_{i \to \infty} b(i) = 0$ and as a result $\mathcal{L}(K_i)$ converges to the true value of the unknown part of the plant state.

Now, we give an illustrative example showing the loss of privacy about the plant state information.

B. An Illustrative Example

Consider system (1) with input (13) under Assumptions (A1)-(A3). We let the parameters of the system be

$$A = \begin{bmatrix} 1 & 0.0022 \\ -0.0044 & 1.0066 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0.0022 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

$S = [1 \;\; -4]$ and $\mu = 49.0636$. This value of μ ensures that the inter-event times are larger than $n + 1 = 3$. In this example $\mathcal{X}$ is a line spanned by the vector $(1, 0)$. We consider the initial plant state $x(0) = (0.8, -0.4)$ and notice that $x_{\zeta}(0) = (1.1198, 0)$.

The evolution of the known and unknown part of plant-states in the event-triggered feedback stabilization task is shown in Figure 2. This verifies the results in Theorem 2 and Corollary 2. Also note that $x^r(k)$ approaches negative of $x^{\zeta}(k) = x^{\zeta}(0)$ asymptotically. Further, the uncertainty sets $\mathcal{L}(K_i)$ are intervals of the line $\mathcal{X}$. In Figure 3, we show the evolution of the left and the right ends of the intervals $\mathcal{L}(K_i)$. We can see here that the length of the line segments $\mathcal{L}(K_i)$

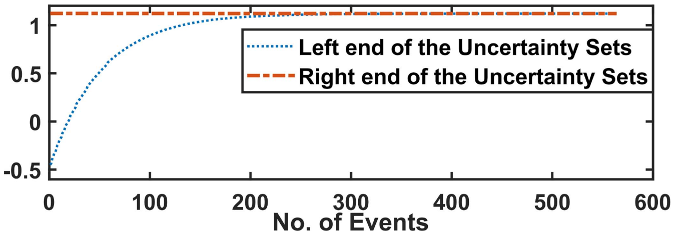

Fig. 3. The left and the right ends of the intervals of $\mathcal{X}$ that are the uncertainty sets $\mathcal{L}(K_i)$. In this figure, we see that the size of the uncertainty set, i.e., $|\mathcal{L}(K_i)|$ is bounded and decreasing with events.

reduces with events. Also note here that no initial estimate needs to be provided for the uncertainty set. Hence, we can see that ED can identify the unknown part of the plant state within a quantifiable bound even in finite time. Moreover, the bound shrinks with each event and converges to zero asymptotically. Thus, in this example, with the knowledge of only the system parameters A, B and C, sensor measurements, the event times and the event-triggering rule, ED is able to breach the privacy of the plant state.

VI. Conclusion

In this letter, we characterized the unobservable subspace of discrete-time LTI systems under unknown piece-wise constant inputs when the input update times are known. In particular, we showed that if the input inter-update times are long enough and if there are enough measurements in each inter-update interval, then the unobservable subspace remains fixed. We then explored the consequences of this result for privacy in event-triggered control. We showed that if an eavesdropper knows the system matrices, the input update times and the event-triggering rule then it can estimate the plant state up to a bound that decreases with time to zero.

Acknowledgment

The authors thank the anonymous reviewers for their constructive comments, which helped improve this letter.

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