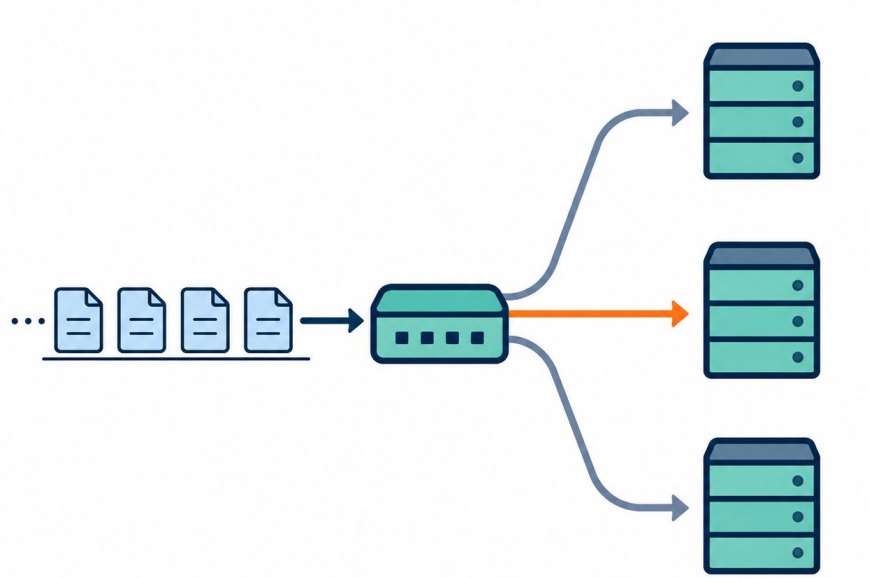
Finite-Time Behavior of Erlang-C Model

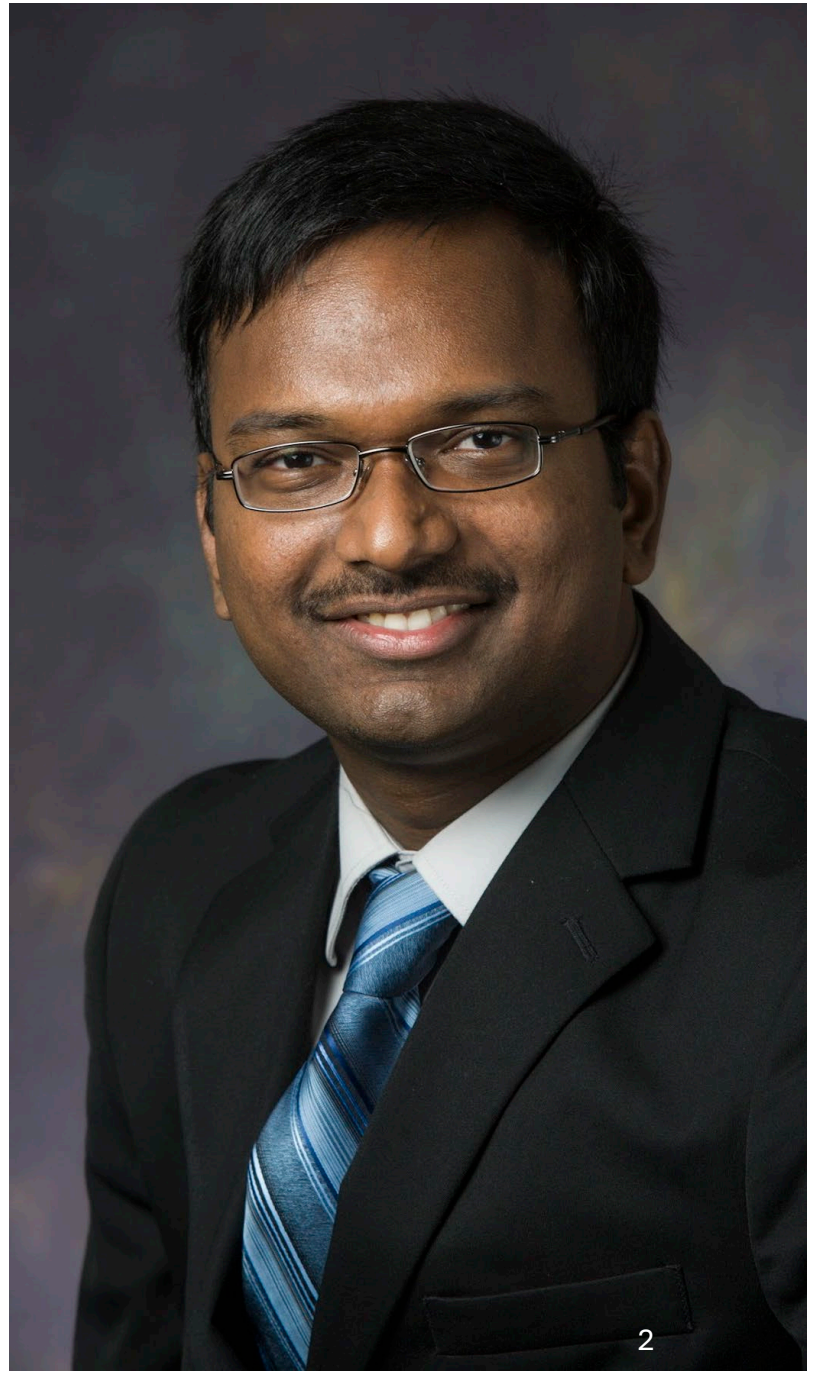
Mixing Time, Mean Queue Length, and Tail Bounds



Siva Theja Maguluri
Industrial and Systems Engineering
Georgia Institute of Technology









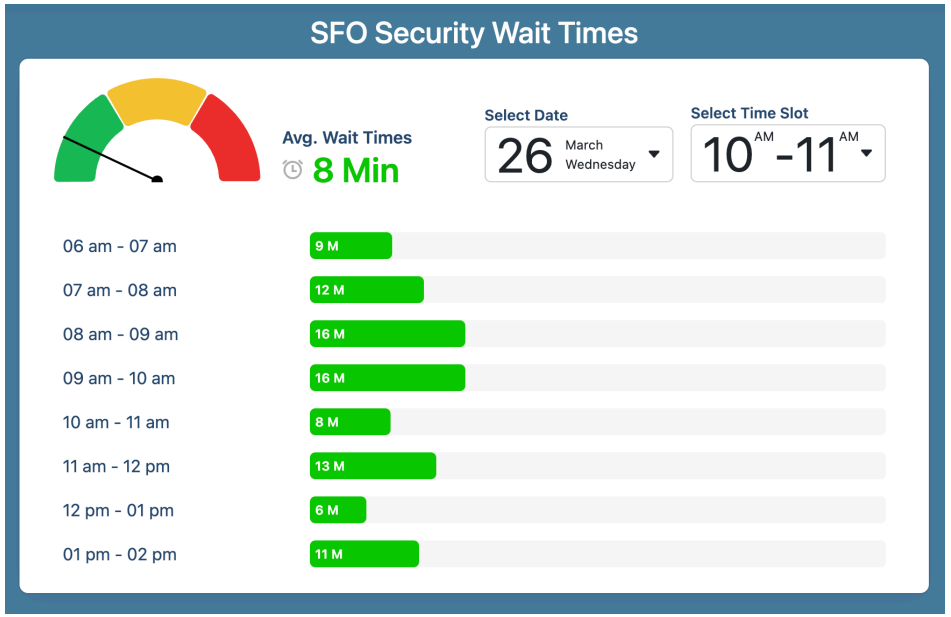
Medical Service Providers



Ride-hailing Platform



Computing Systems



Average wait time



Medical Service Providers



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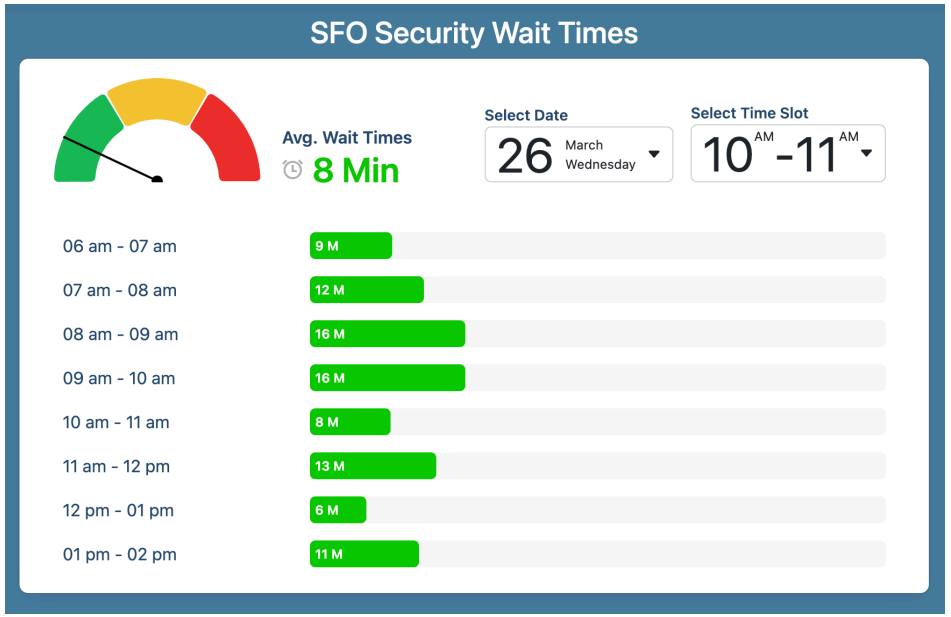

Average wait time



Service level guarantee



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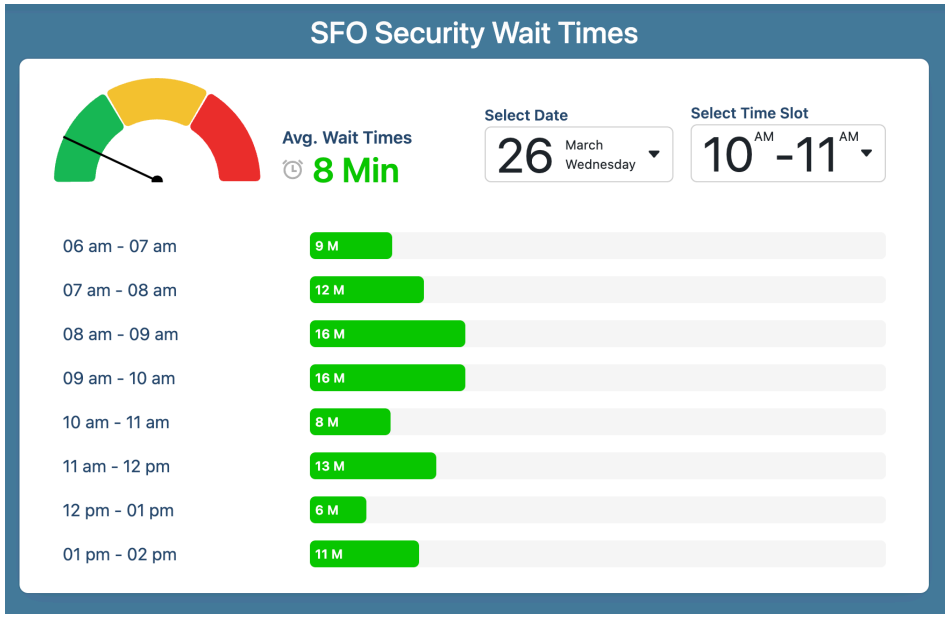


Average wait time

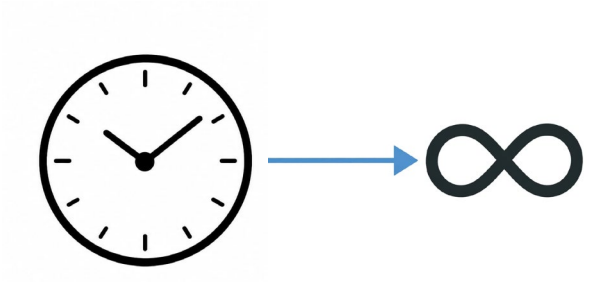
$\mathbb{E}[\mathbf{q}]$



Service level guarantee

$\mathbb{P}[\mathbf{q} > \mathbf{x}]$

Asymptotics

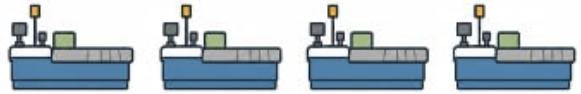


Steady-state approximation



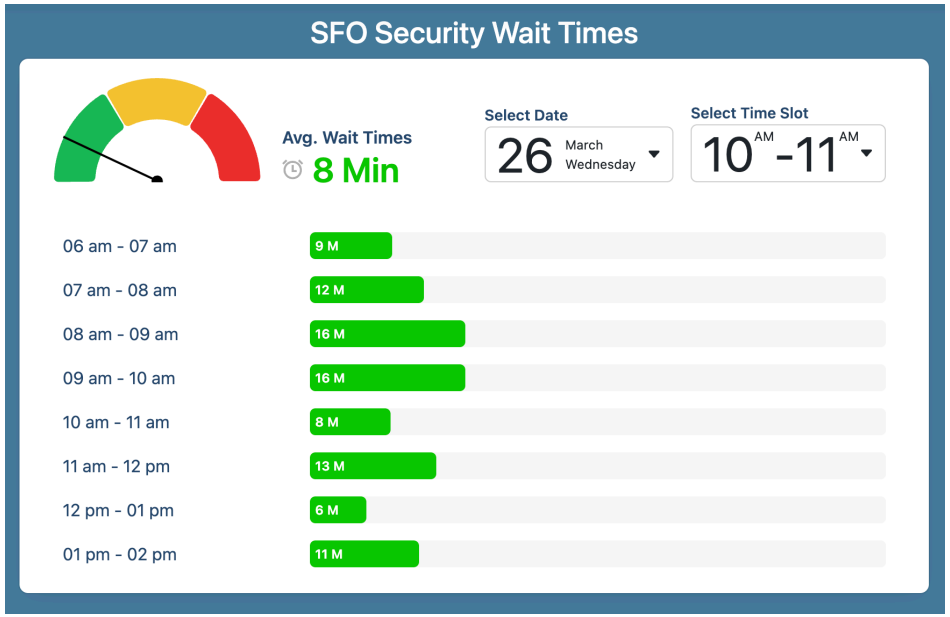
$\lambda \to \mu$

Heavy-traffic approximation



$n \to \infty$

Many-server asymptotic



Average wait time

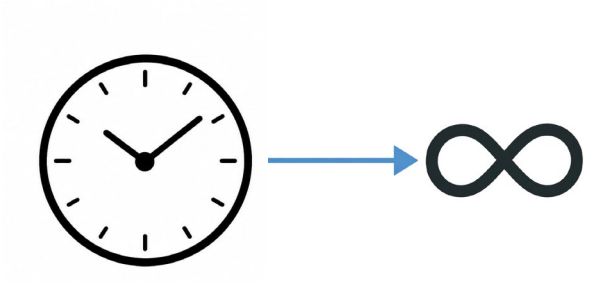
$\mathbb{E}[\mathbf{q}]$



Service level guarantee

$\mathbb{P}[\mathbf{q} > \mathbf{x}]$

Our goal



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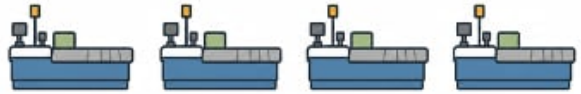

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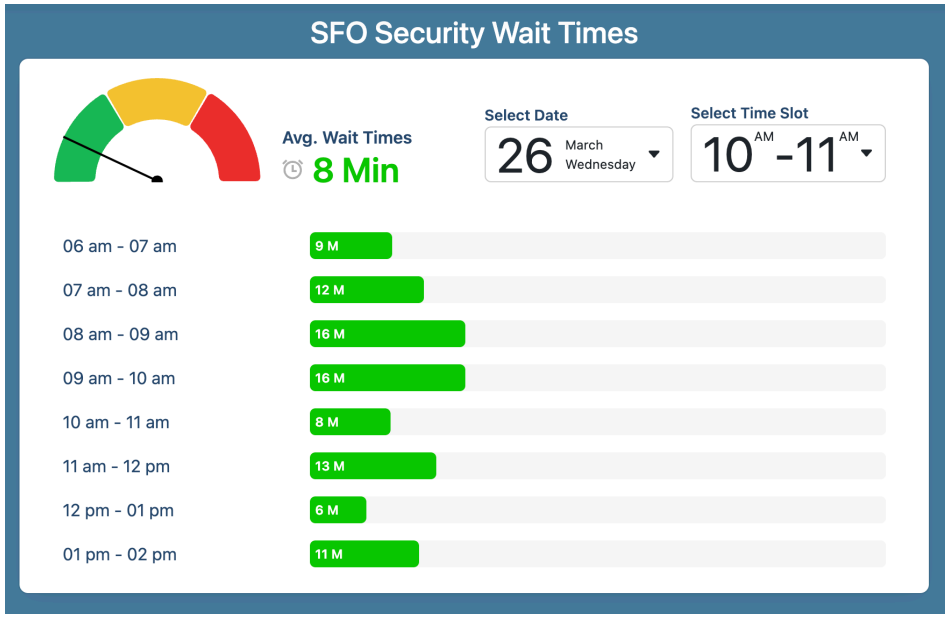

Average wait time

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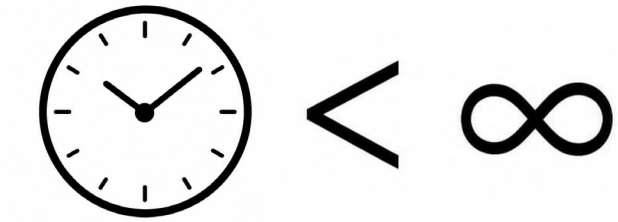
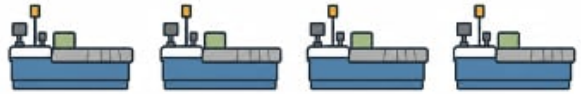


Finite time



$\lambda \to \mu$

Heavy-traffic approximation



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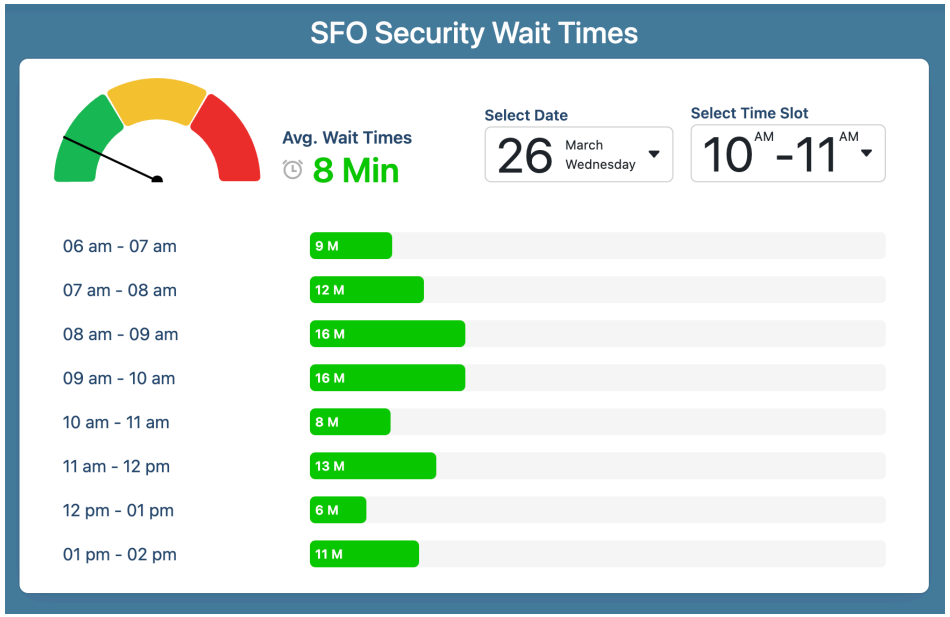
Many-server asymptotic



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$\mathbb{E}[\mathbf{q}]$



Service level guarantee

$\mathbb{P}[\mathbf{q} > \mathbf{x}]$

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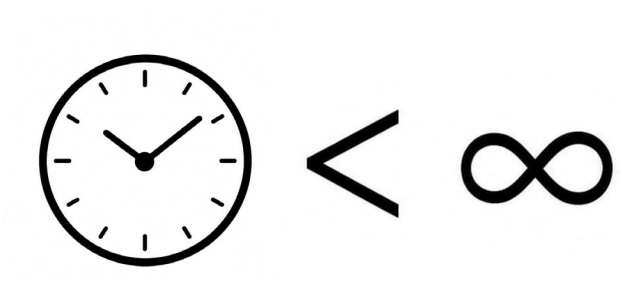
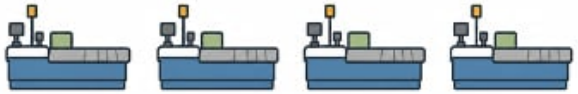


Finite time



$\lambda < \mu$

Non-asymptotic load



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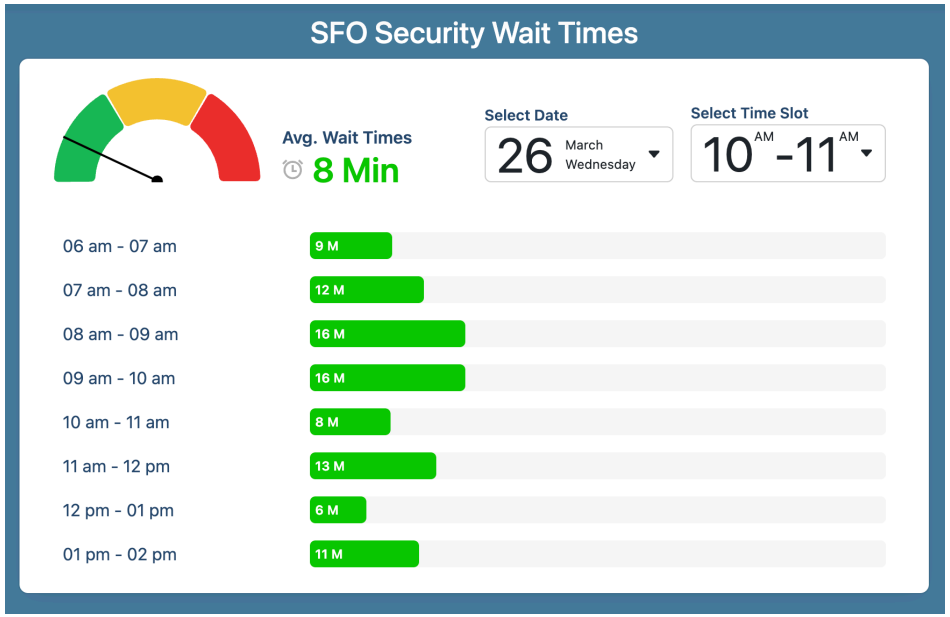
Many-server asymptotic



Average wait time

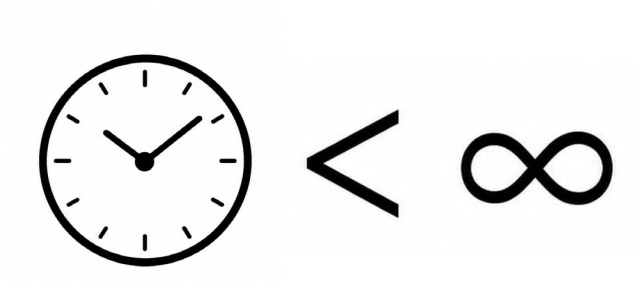
$\mathbb{E}[\mathbf{q}]$



Service level guarantee

$\mathbb{P}[\mathbf{q} > \mathbf{x}]$

Our goal



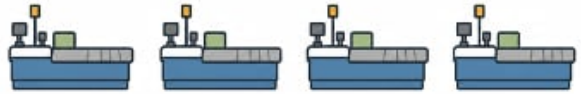
$< \infty$

Finite time



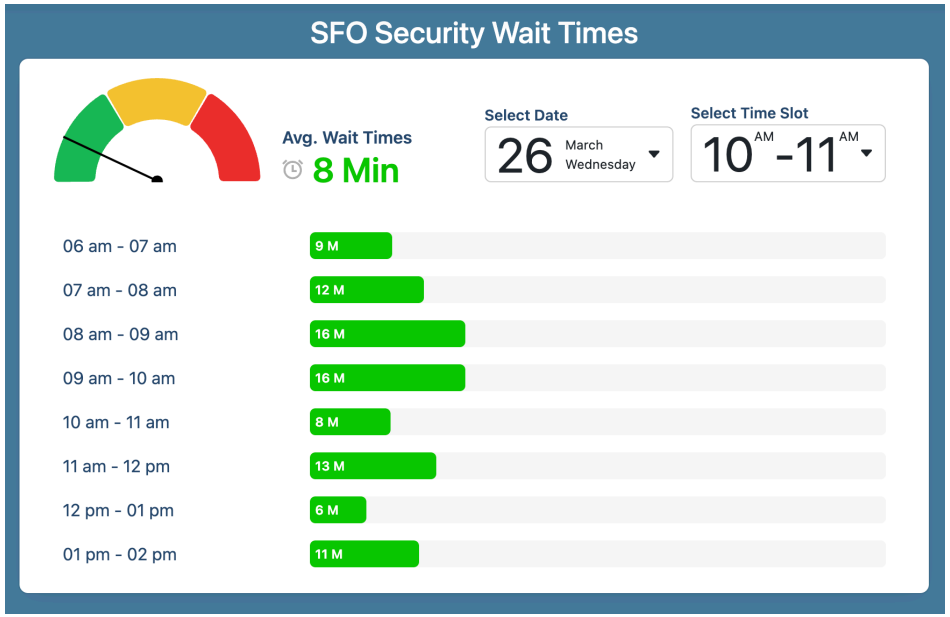
$\lambda < \mu$

Non-asymptotic load



$n < \infty$

Finite size



Average wait time

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Service level guarantee

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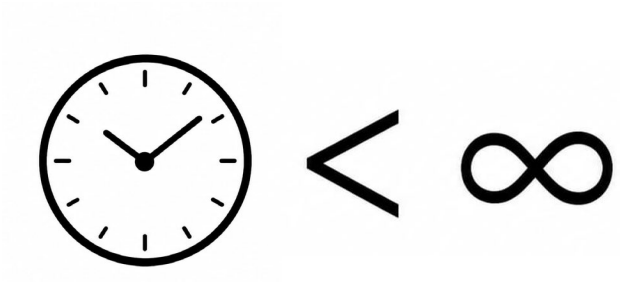
Our goal



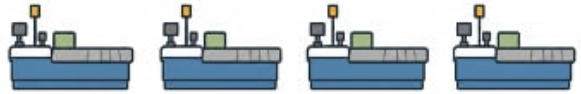
$< \infty$

Finite time



$\lambda < \mu$

Non-asymptotic load



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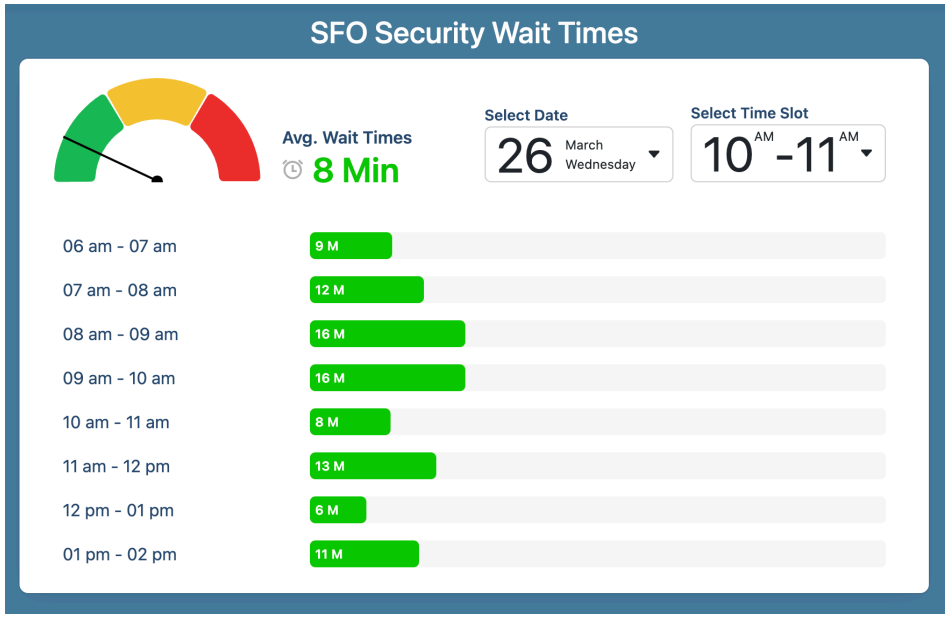
Finite size



Average wait time

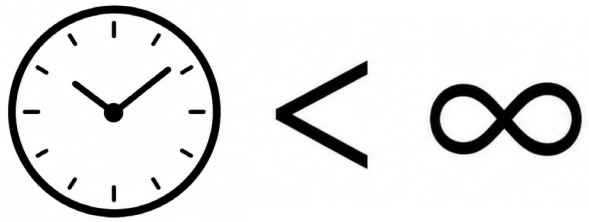
$\mathbb{E}[\mathbf{q}(t)]$



Service level guarantee

$\mathbb{P}[\mathbf{q}(t) > \mathbf{x}]$

Our goal



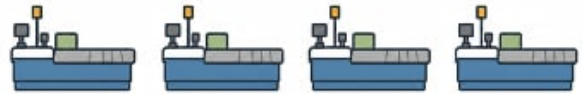
$< \infty$

Finite time



$\lambda < \mu$

Non-asymptotic load



$n < \infty$

Finite size

Finite-time bounds via mixing

$$d_\chi(q(t), q(\infty)) \leq e^{-\mathfrak{Rate} \times t} d_\chi(q(0), q(\infty))$$

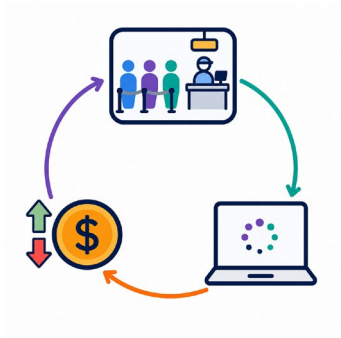
Average wait time

$\mathbb{E}[\mathbf{q}(t)]$

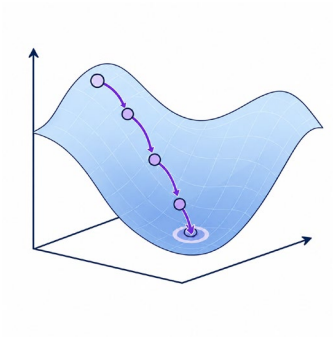
Service level guarantee

$\mathbb{P}[\mathbf{q}(t) > \mathbf{x}]$

Another Motivation: Learning for service systems



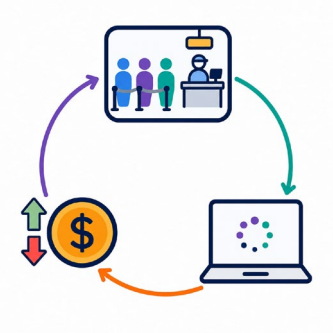
Collect samples



Pricing/Staffing Policy

SGD-like update step

Another Motivation: Learning for service systems



Collect samples

Pricing/Staffing Policy

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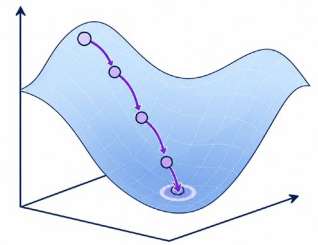
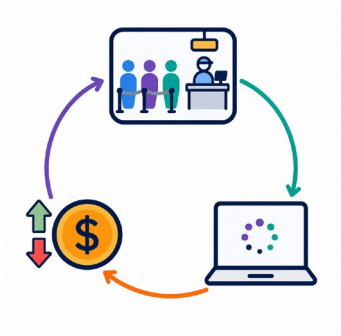
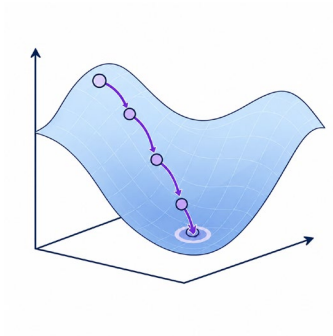


Update policy

Another Motivation: Learning for service systems





Collect samples

Pricing/Staffing Policy

SGD-like update step

Update policy

Mixing times are needed to obtain finite-time regret and sample complexity bounds

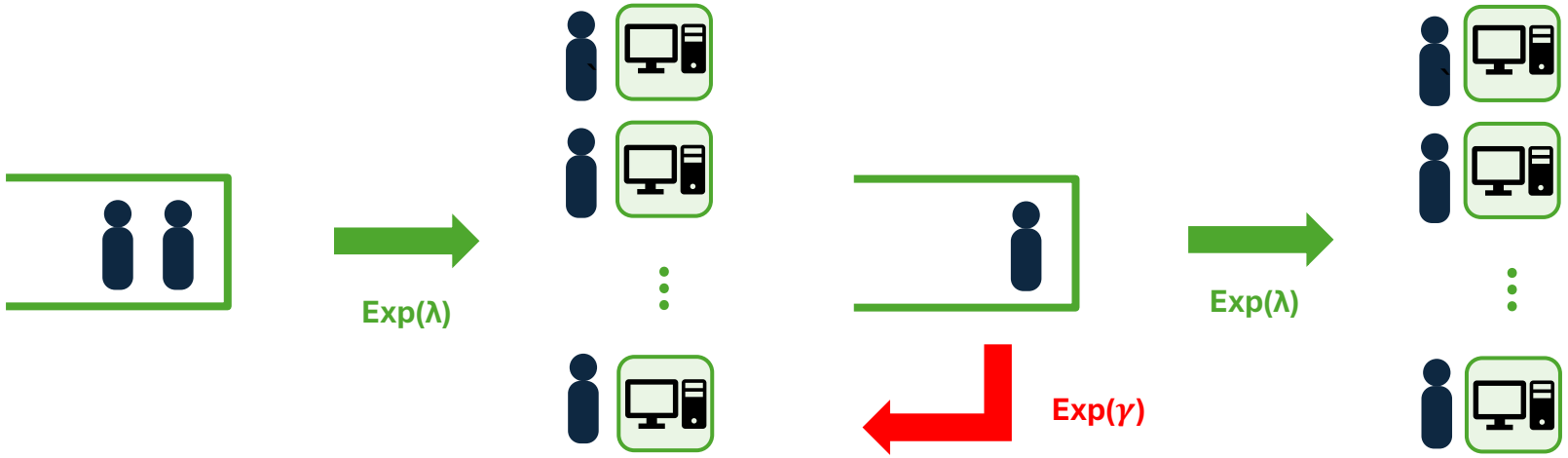
Overview of our work

Multiserver queue

[SIGMETRICS 2025, OR major revision]

Abandonment queue

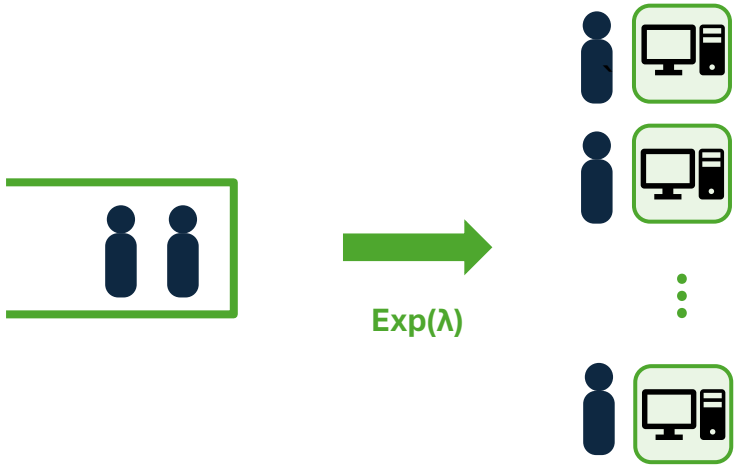
Forthcoming



Overview of our work

Multiserver queue

[SIGMETRICS 2025, OR major revision]



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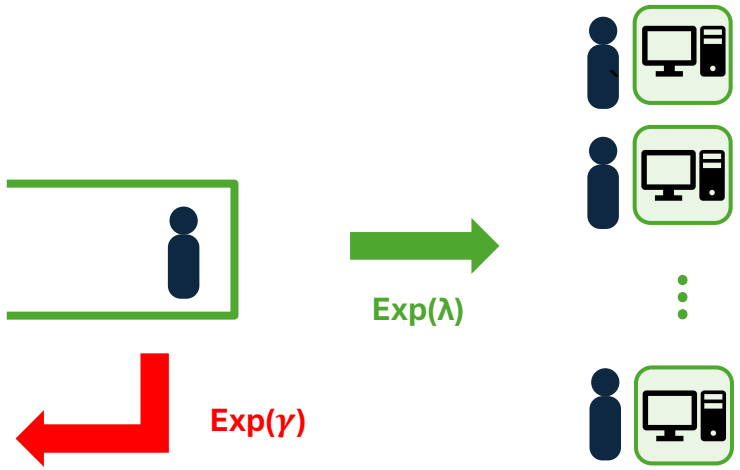
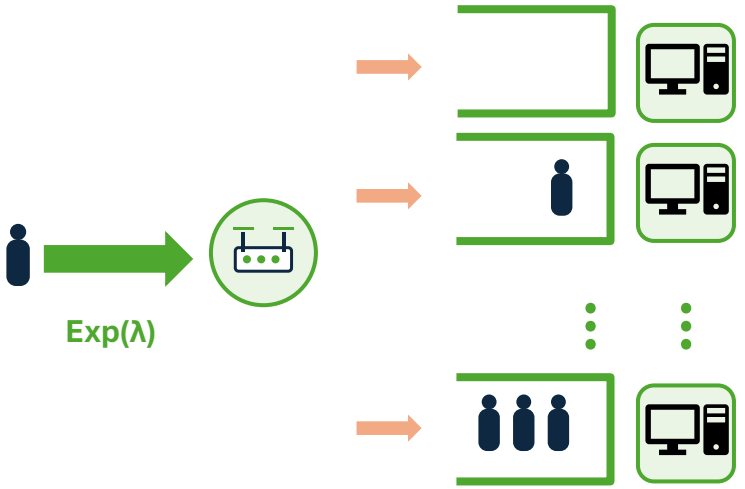


Load-balancing system

[JAP 2026], Ongoing



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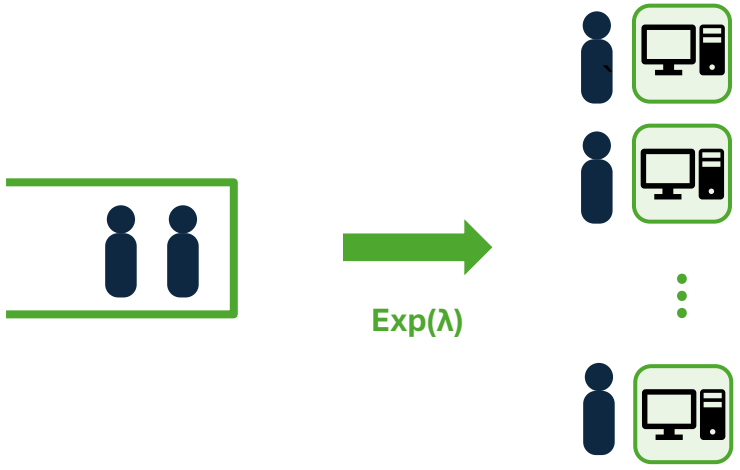

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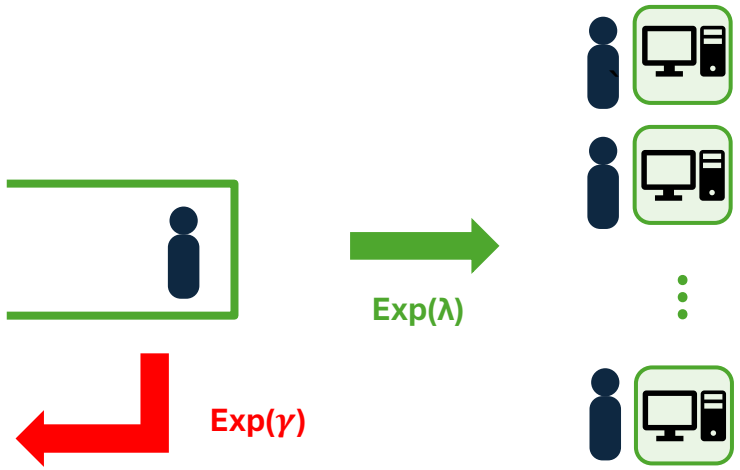

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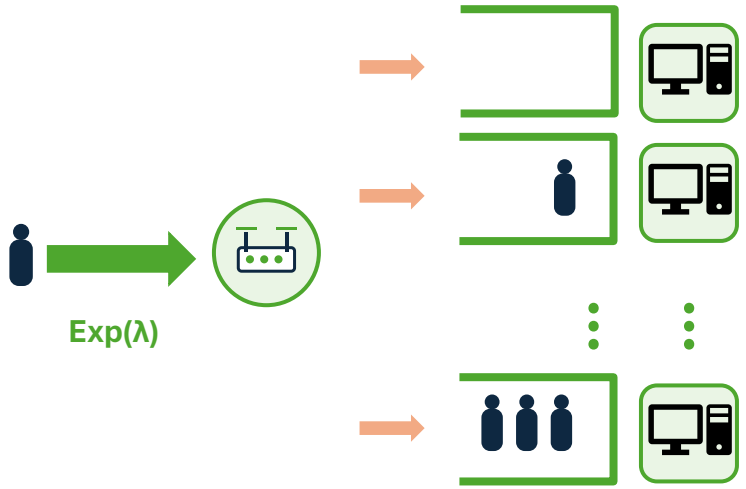

Key Challenges

Overview of our work

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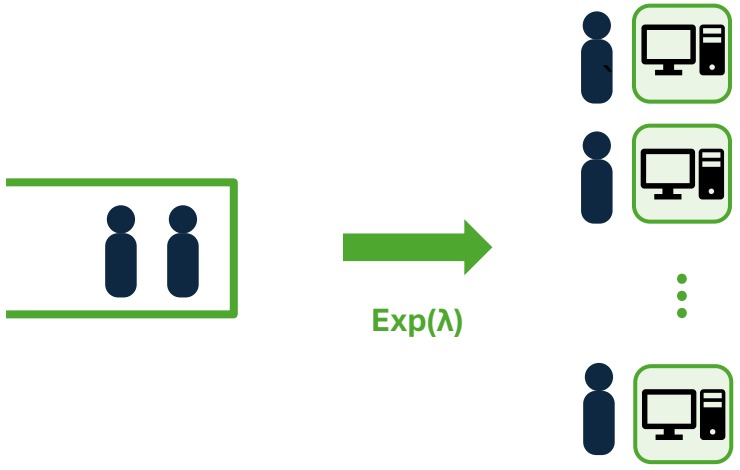

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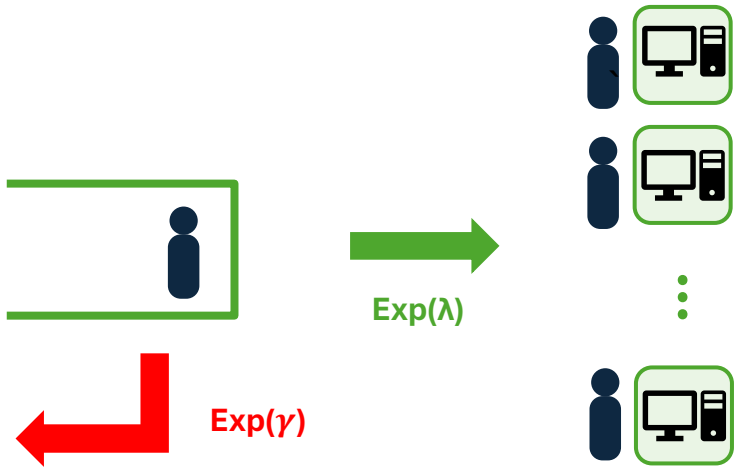

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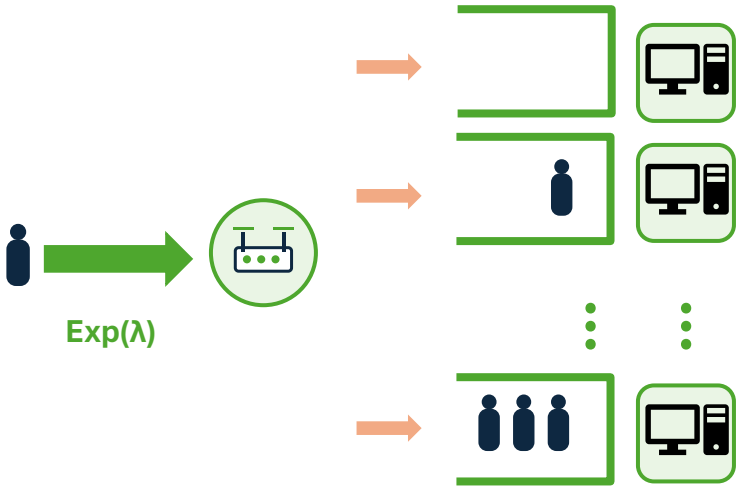
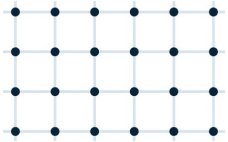


Key Challenges

Infiniteness



Discreteness



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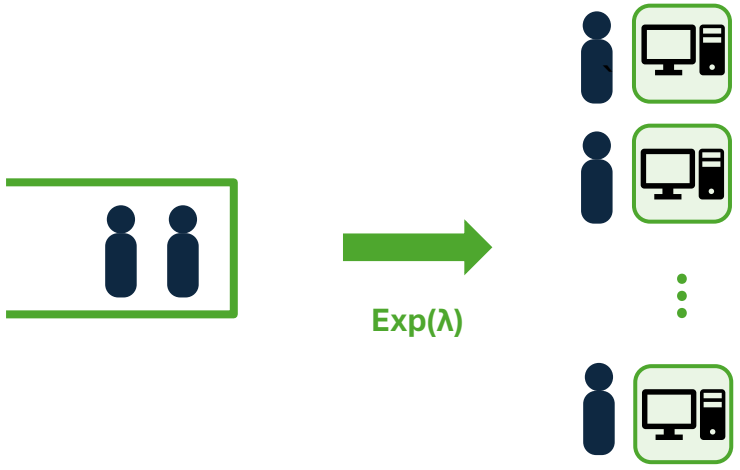

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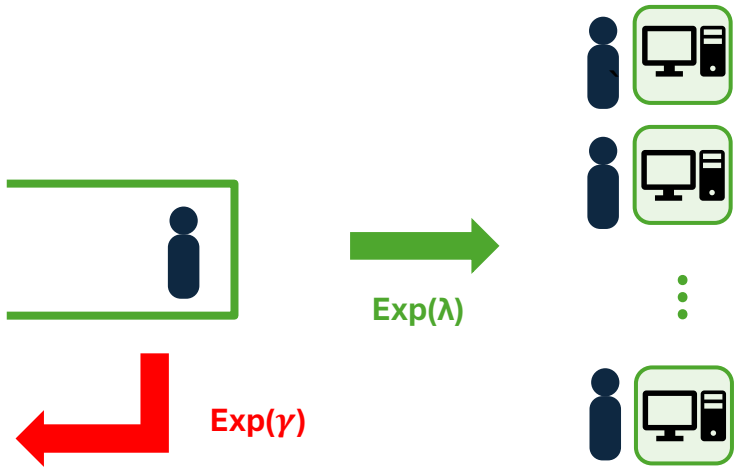

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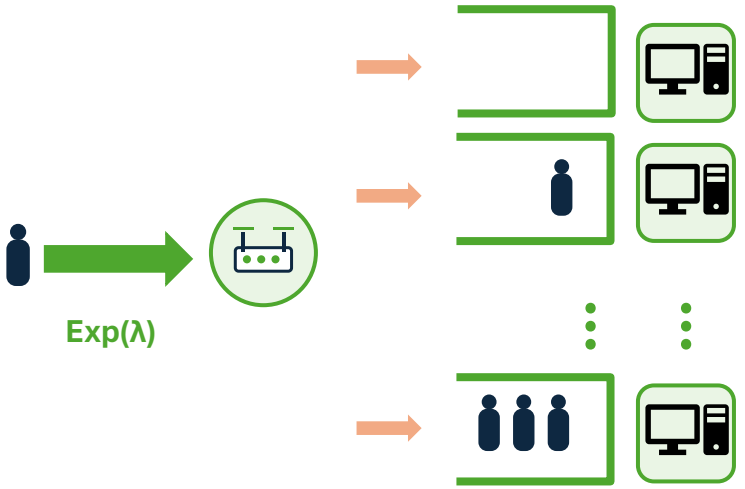

Lyapunov drift method

Key Challenges

Infiniteness



Discreteness



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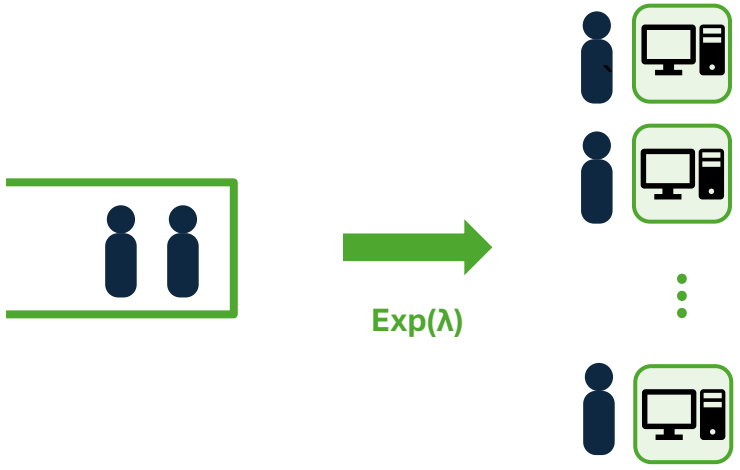

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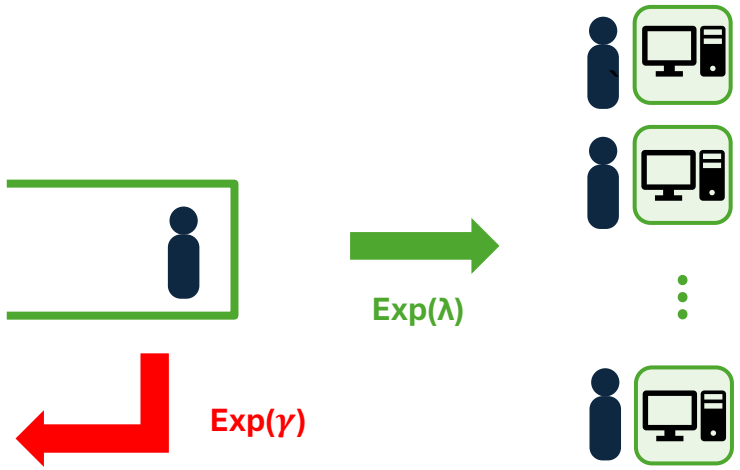
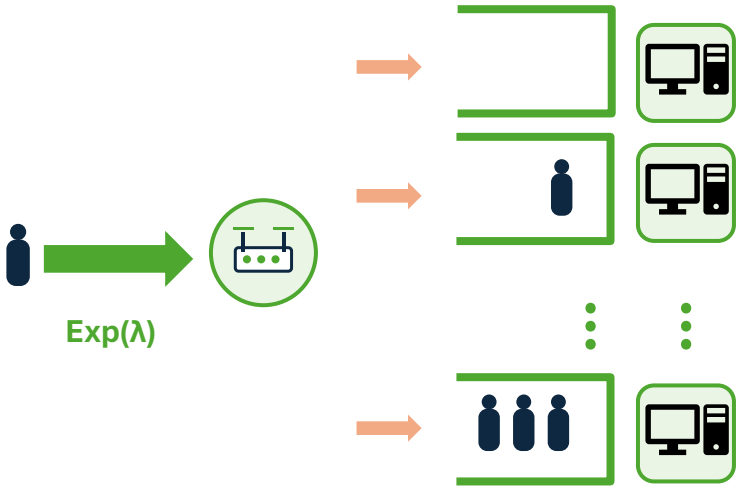


Load-balancing system

[JAP 2026], Ongoing



Lyapunov drift method

Curvature method

Key Challenges

Infiniteness



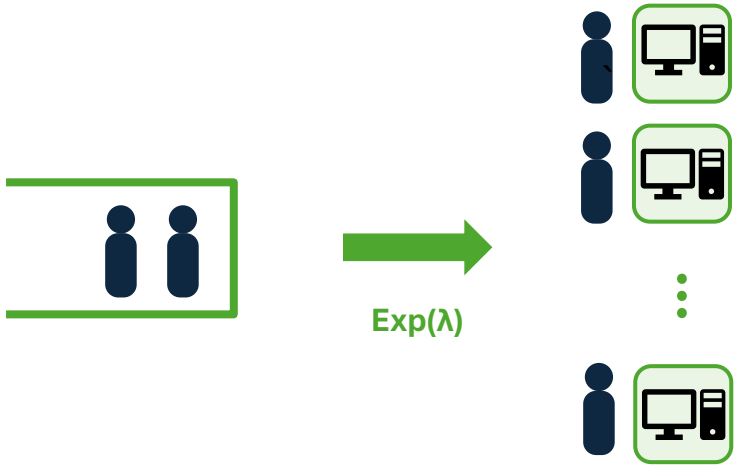
Discreteness



Overview of our work

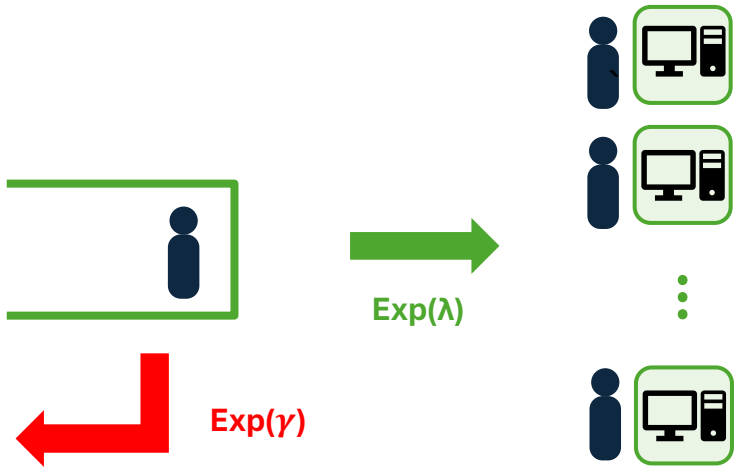
Multiserver queue

[SIGMETRICS 2025, OR major revision]



Abandonment queue

Forthcoming



Load-balancing system

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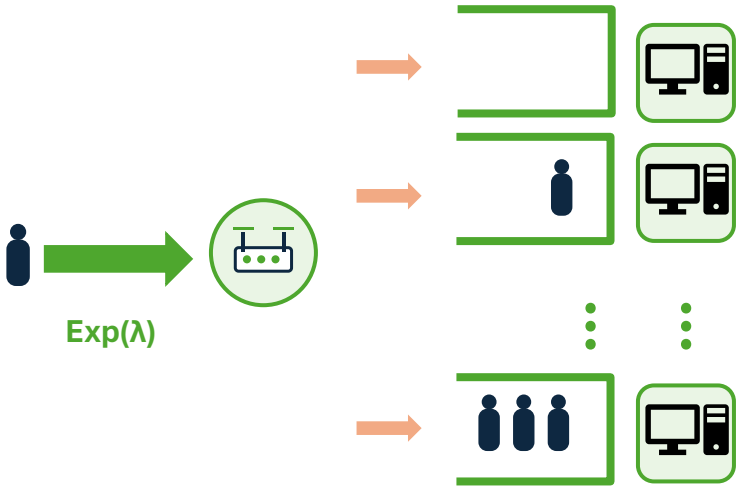


Lyapunov drift method

Curvature method

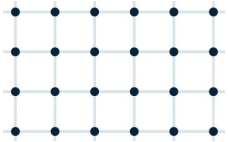
Coupling method

Key Challenges

Infiniteness



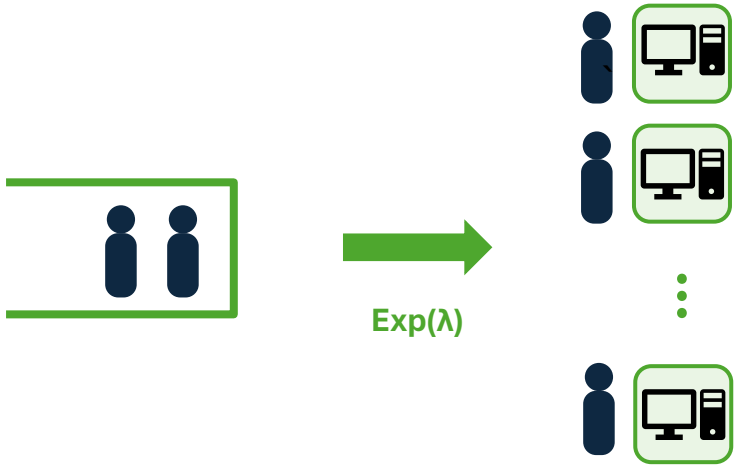
Discreteness



Overview of our work

Multiserver queue

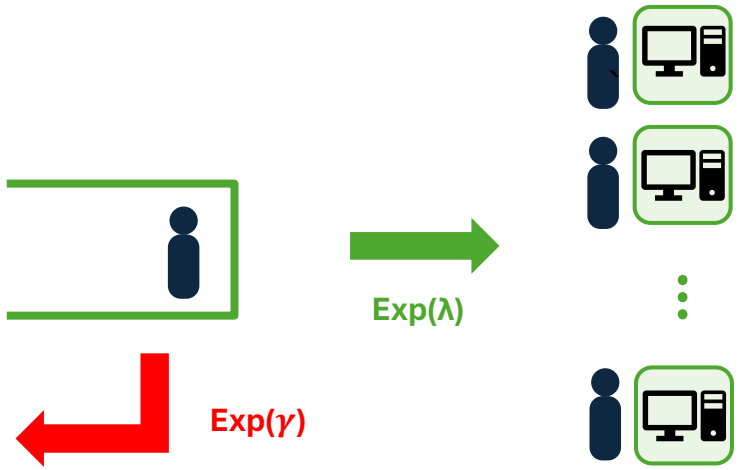
[SIGMETRICS 2025, OR major revision]



Lyapunov drift method

Abandonment queue

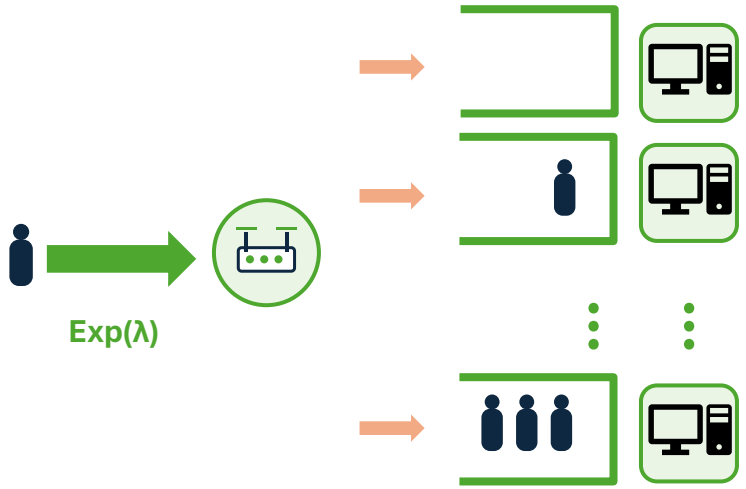
Forthcoming



Curvature method

Load-balancing system

[JAP 2026], Ongoing



Coupling method

Key Challenges

Infiniteness



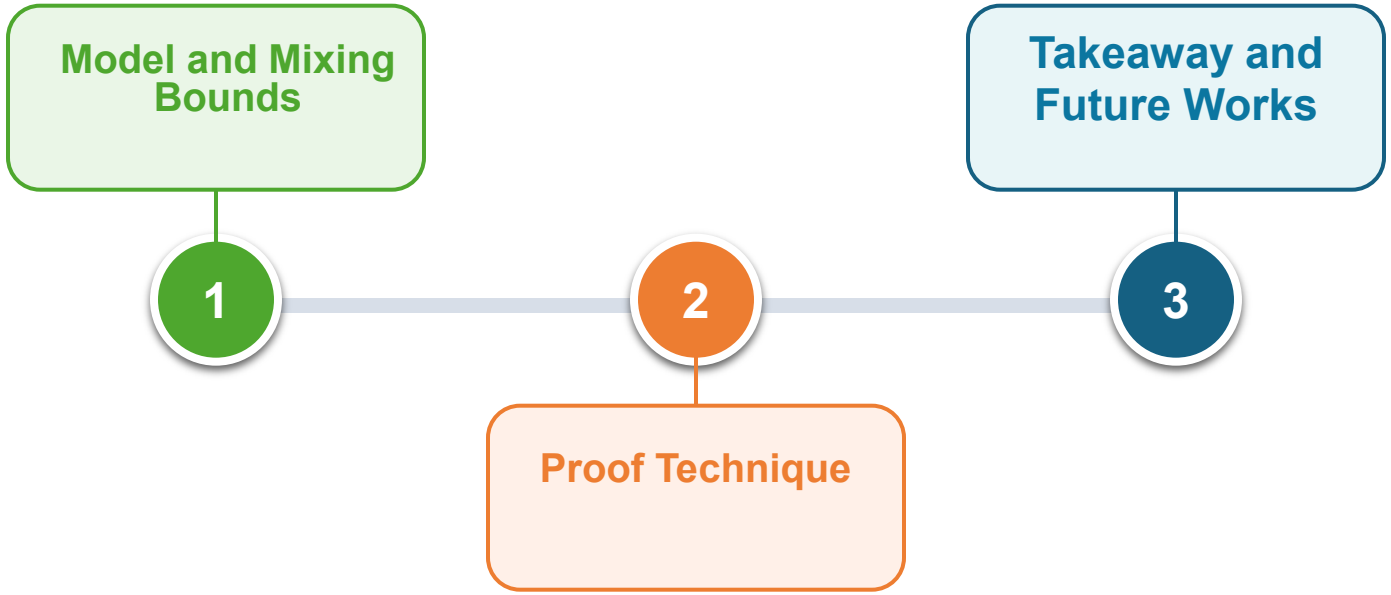
Discreteness



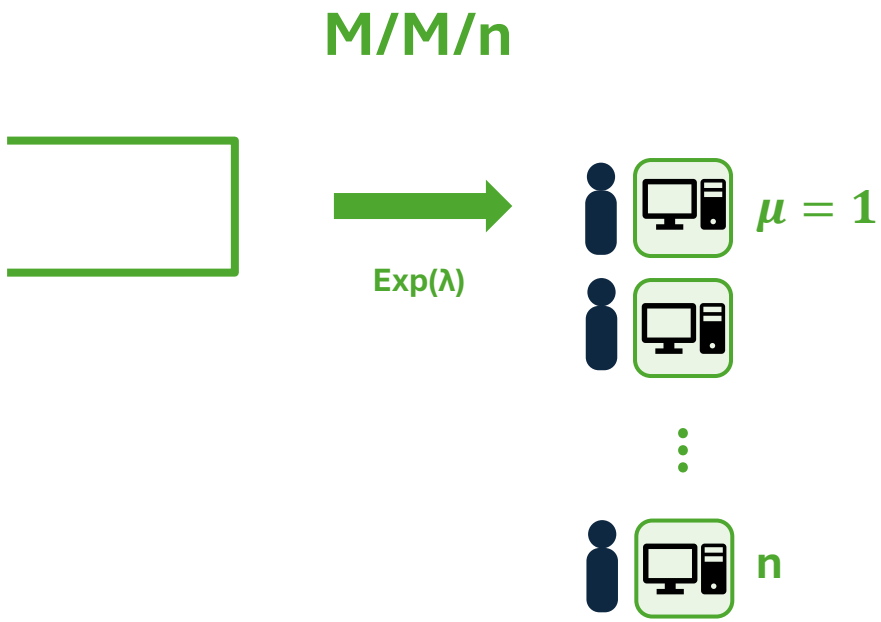
Outline



Outline

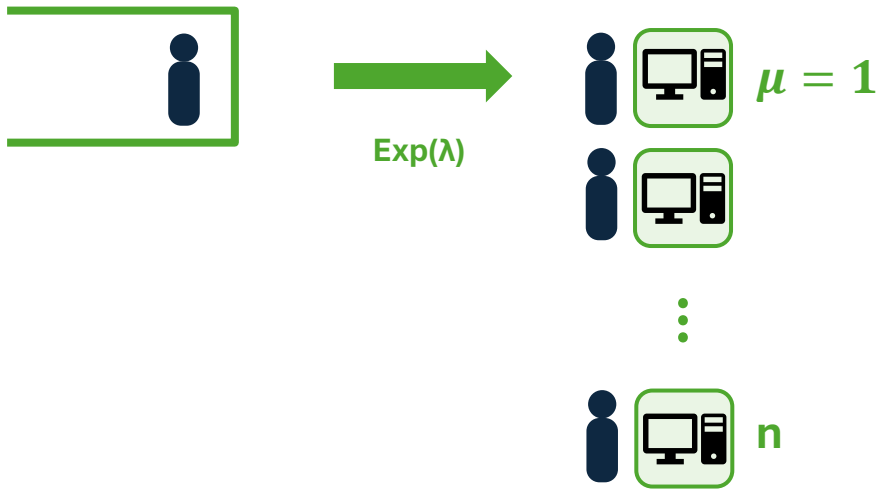

Model and Mixing Bounds
Takeaway and Future Works
1
2
3
Proof Technique

Model: Erlang-C system



Model: Erlang-C system

M/M/n



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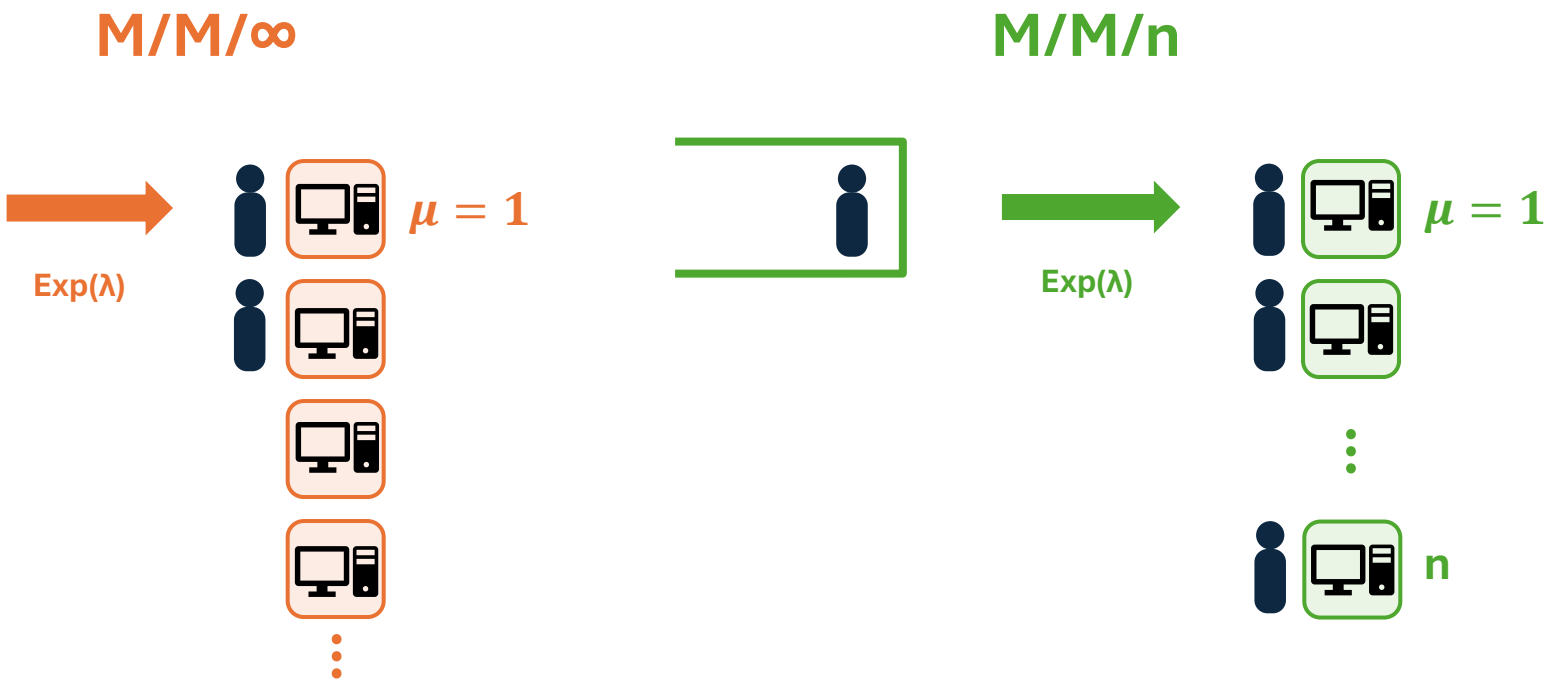

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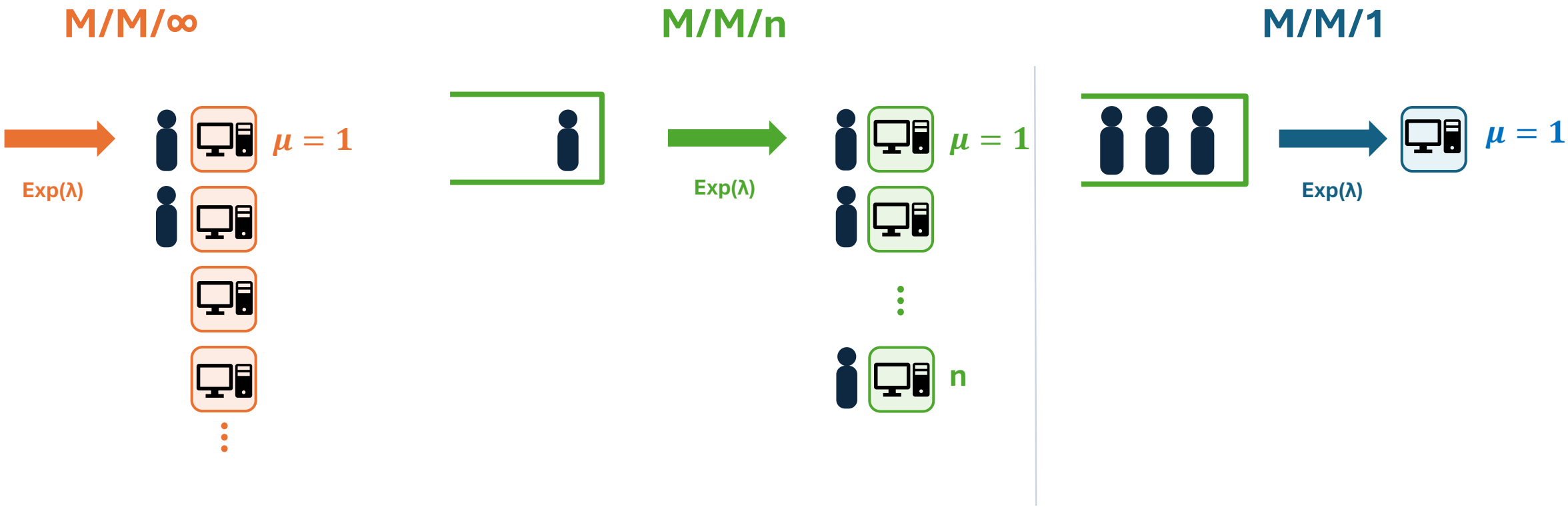

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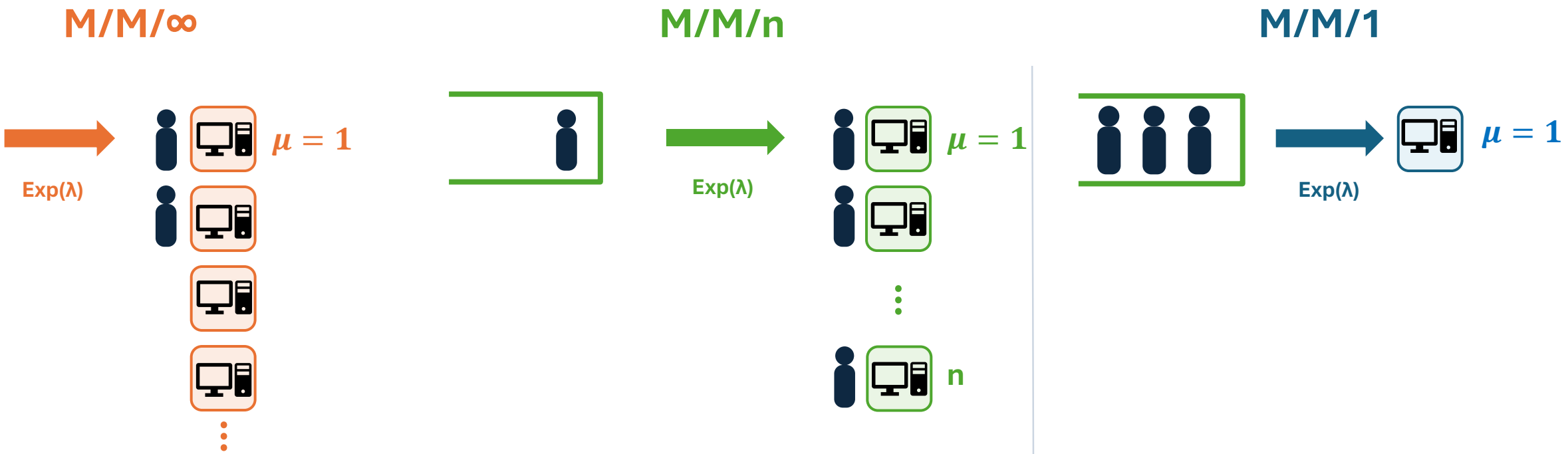
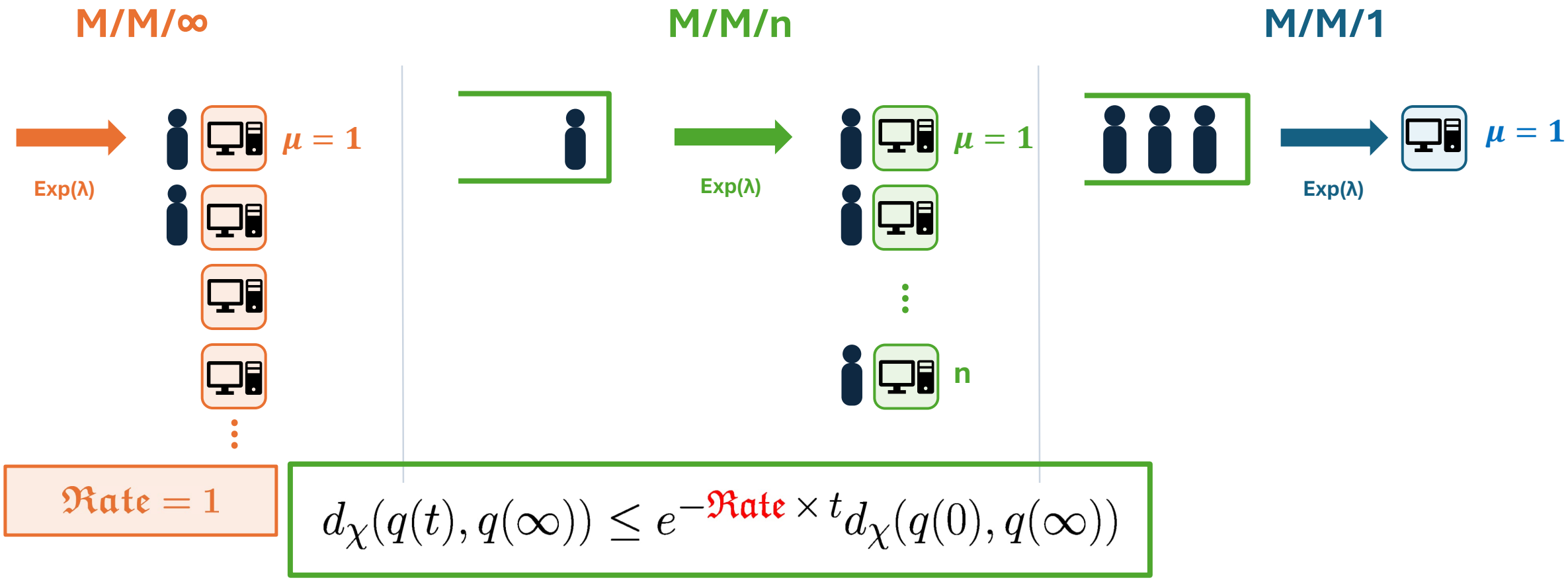


$$d_\chi(q(t), q(\infty)) \leq e^{-\mathfrak{Rate} \times t} d_\chi(q(0), q(\infty))$$

Model: Erlang-C system



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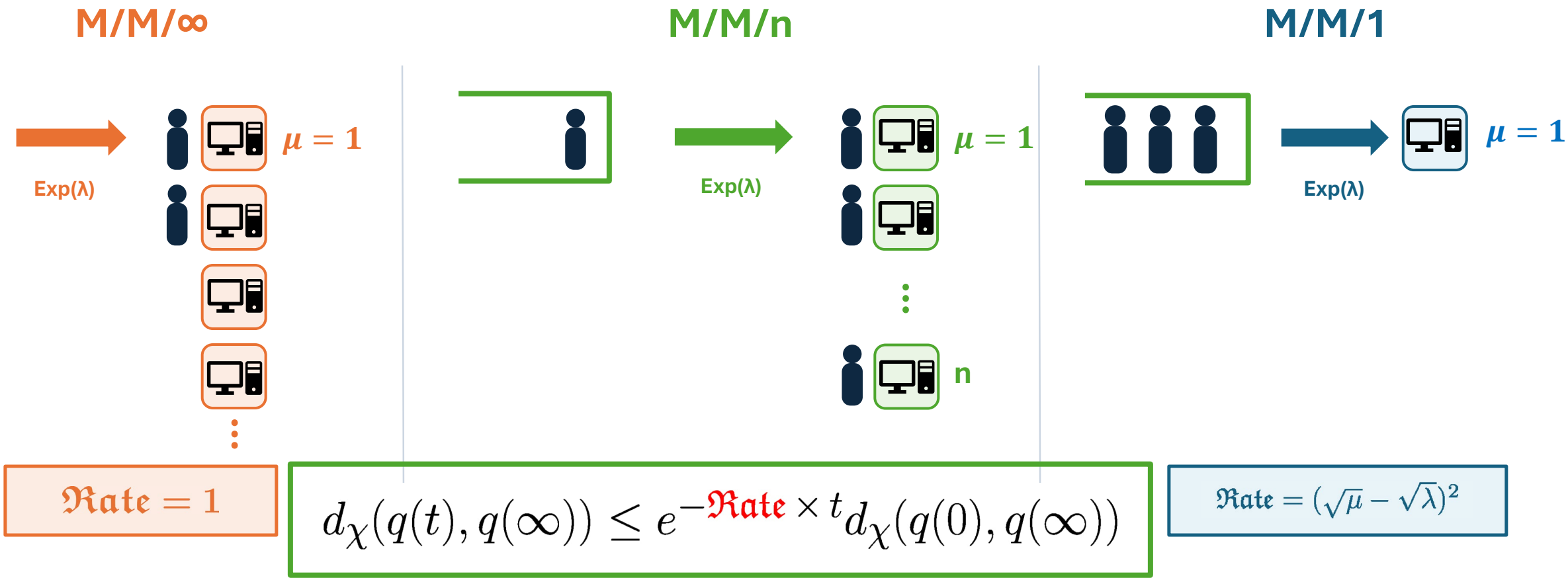
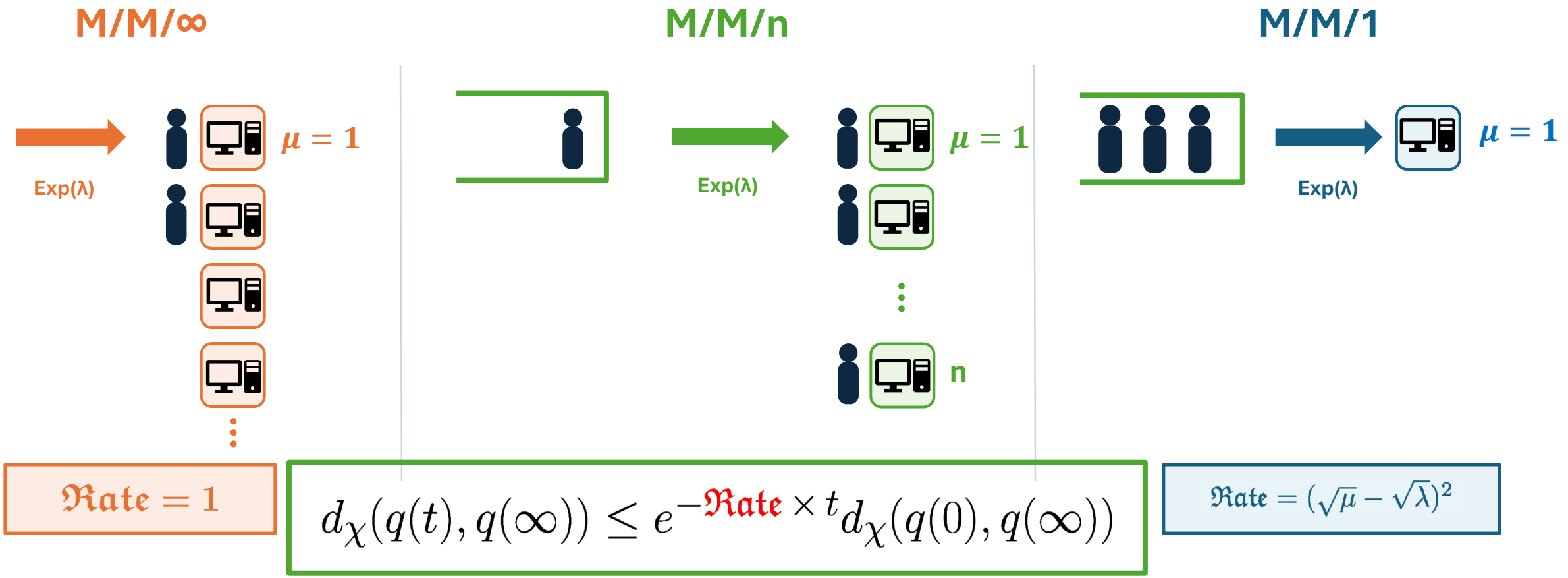


Model: Erlang-C system



$\mathfrak{Rate} = 1$

$$d_\chi(q(t), q(\infty)) \leq e^{-\mathfrak{Rate} \times t} d_\chi(q(0), q(\infty))$$

$\mathfrak{Rate} = (\sqrt{\mu} - \sqrt{\lambda})^2$

Non-asymptotic rate
Theorem 1 [NVM 2025]

$\alpha \in (0,1/2)$

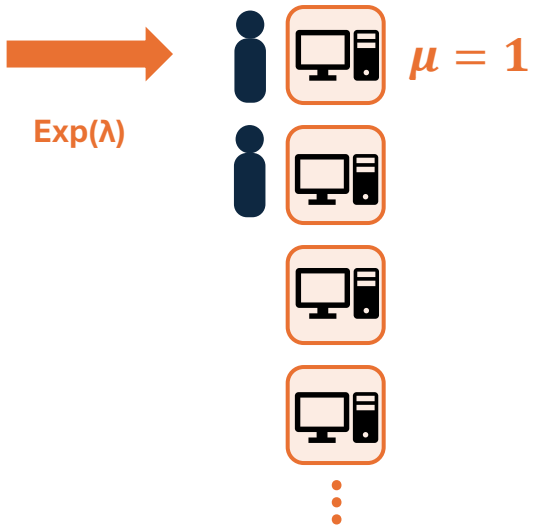
$\mathfrak{Rate} = \Omega(1)$

$\alpha \in (1/2, \infty)$

$\mathfrak{Rate} = C(\sqrt{n} - \sqrt{\lambda})^2$

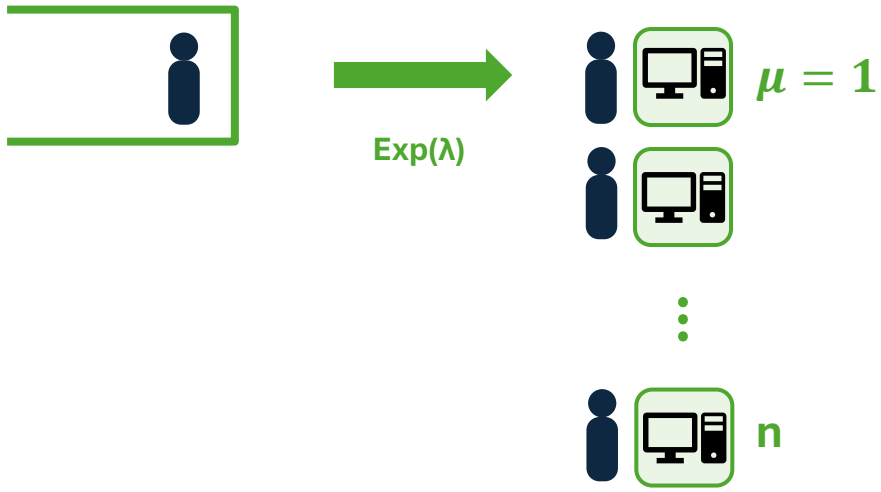
Model: Erlang-C system

M/M/∞

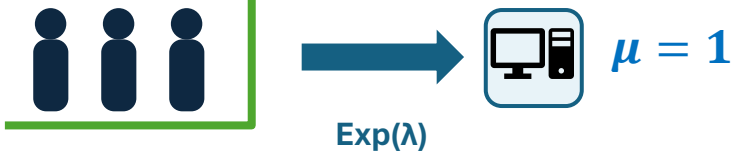


$$\mathfrak{Rate} = 1$$

M/M/n



M/M/1



$$\mathfrak{Rate} = (\sqrt{\mu} - \sqrt{\lambda})^2$$

Non-asymptotic rate
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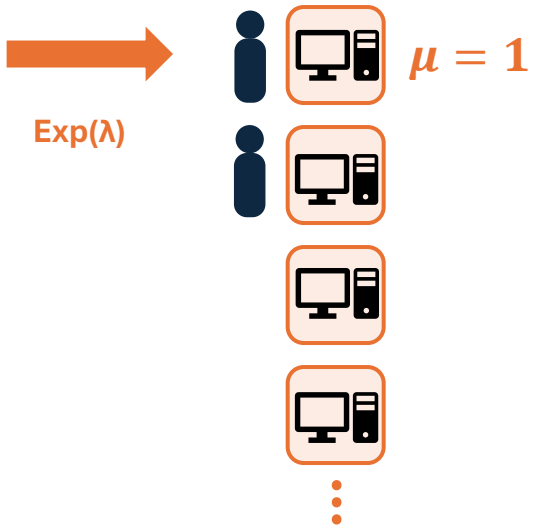

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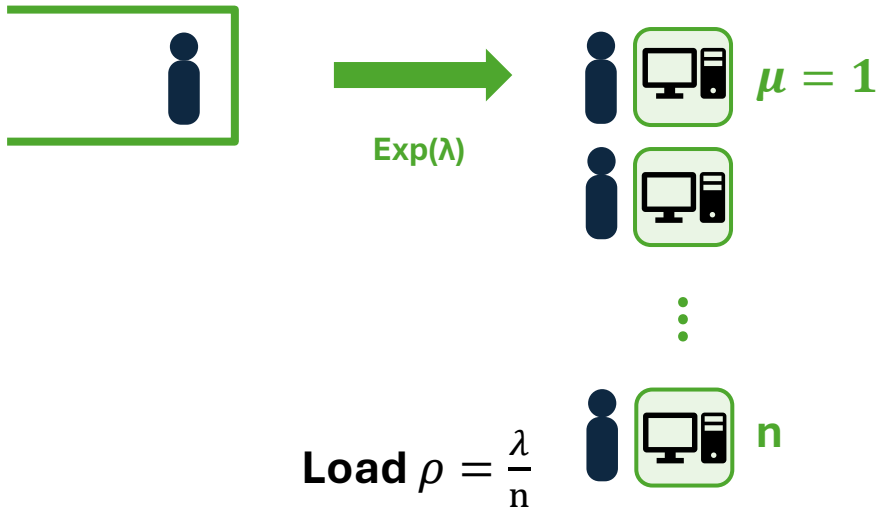
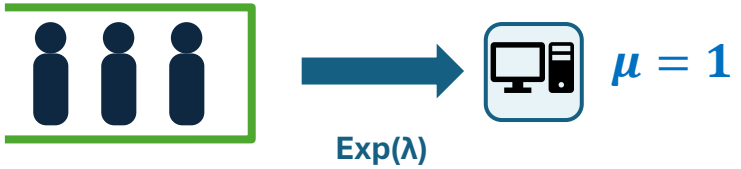


Load $\rho = \frac{\lambda}{n}$

$$\alpha = \frac{\log\left(\frac{1}{1-\rho}\right)}{\log n}$$

M/M/1



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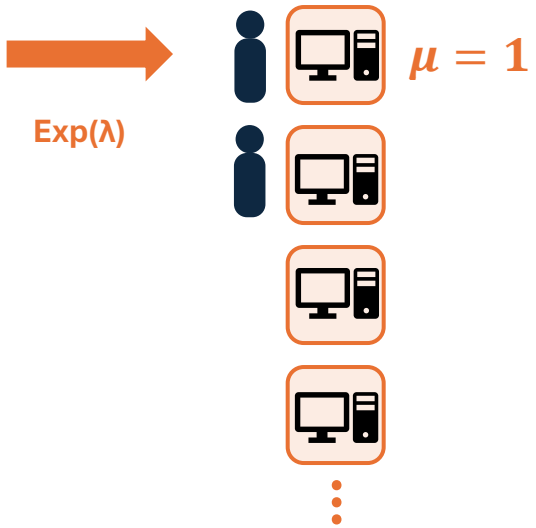

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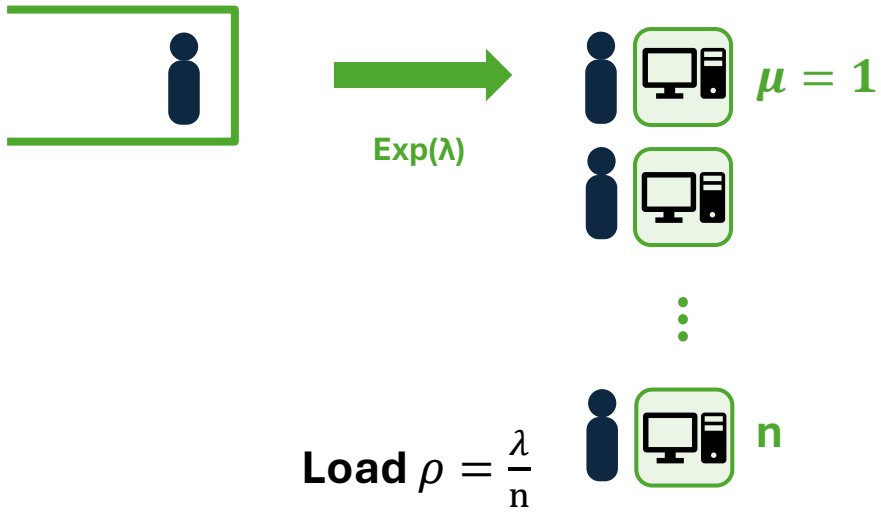
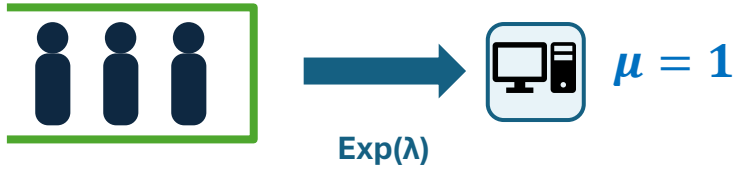


Load $\rho = \frac{\lambda}{n}$

$$\lambda = n - n^{1-\alpha}$$

M/M/1



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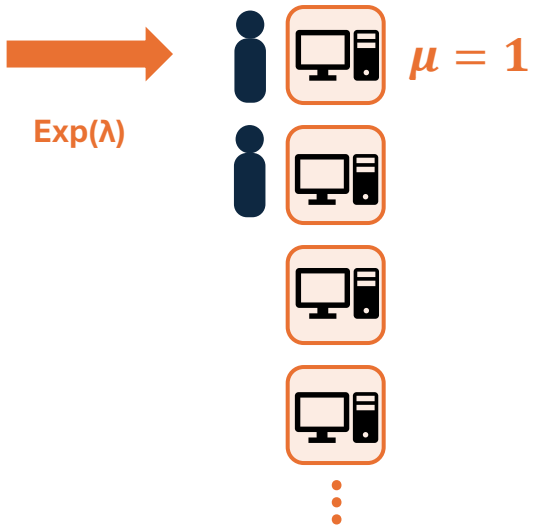

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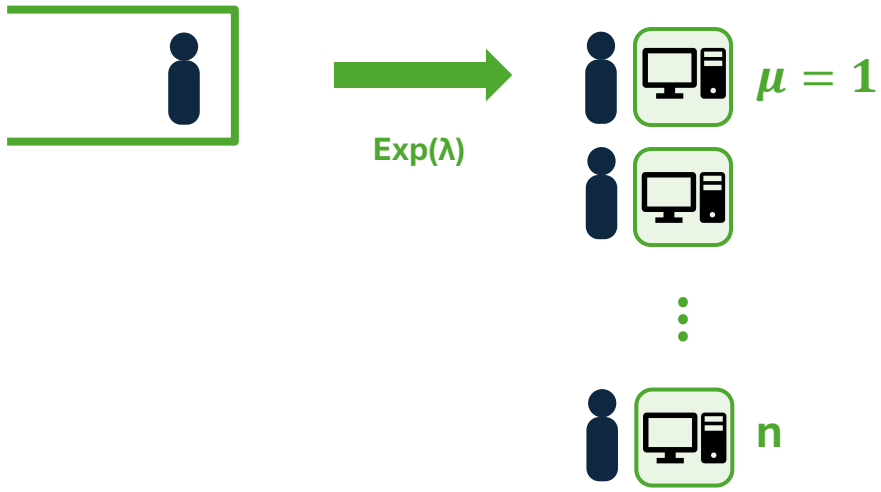

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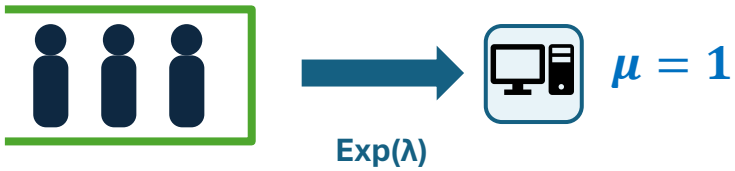

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Sub-Halfin-Whitt

$\alpha \in (0,1/2)$

$$\mathfrak{Rate} = \Omega(1)$$

Halfin-Whitt

$\alpha = 1/2$

Super-Halfin-Whitt

$\alpha \in (1/2, \infty)$

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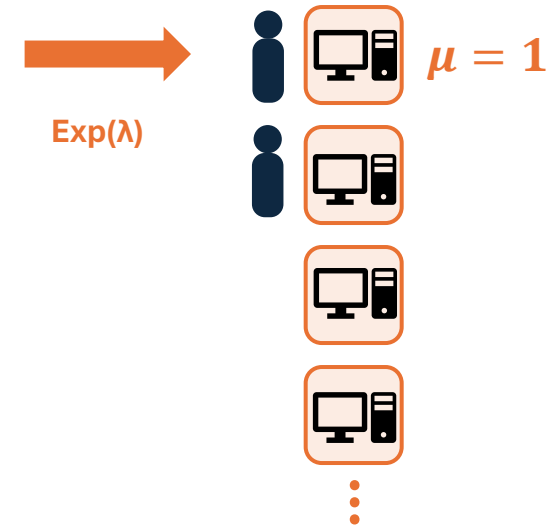
Model: Erlang-C system

M/M/∞



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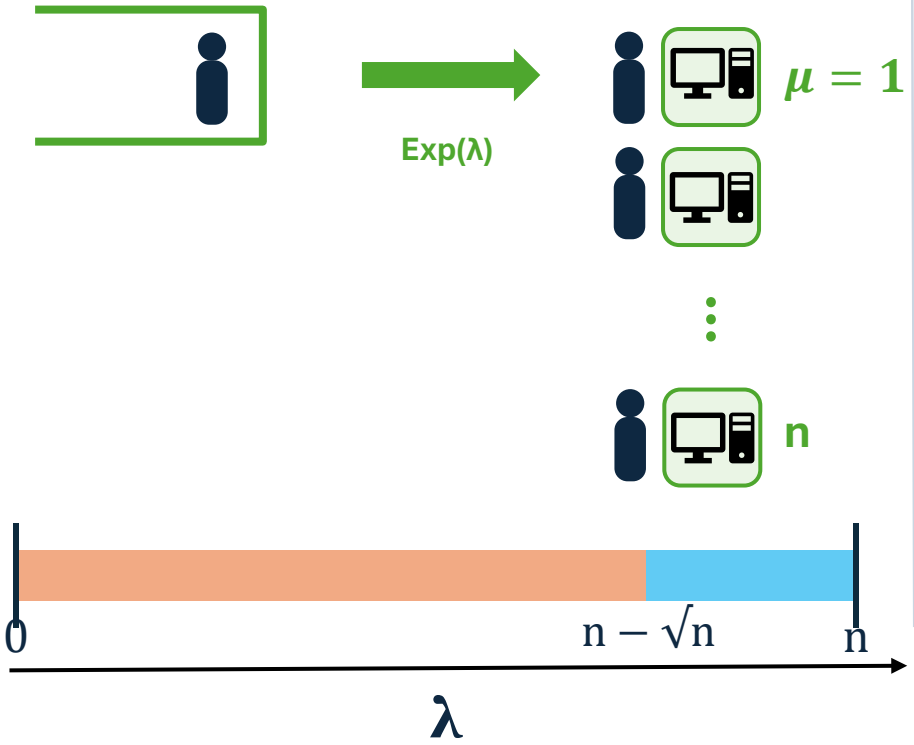

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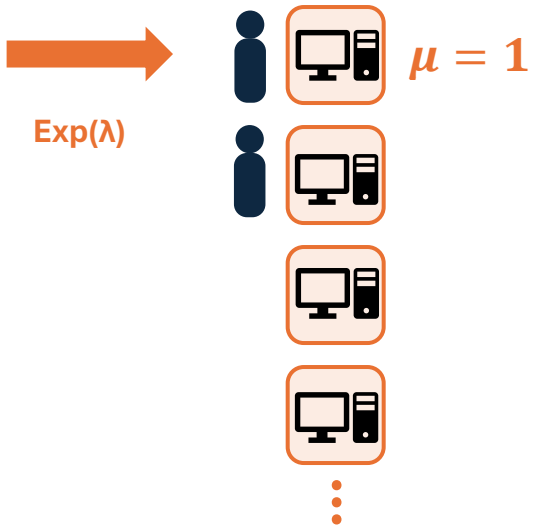

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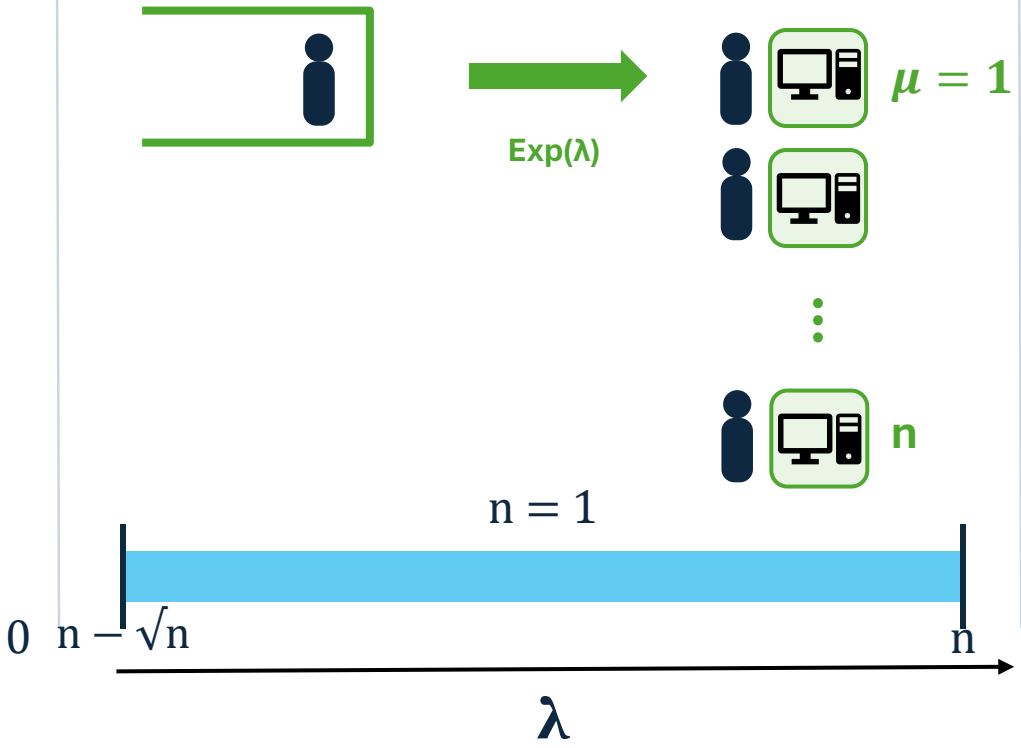

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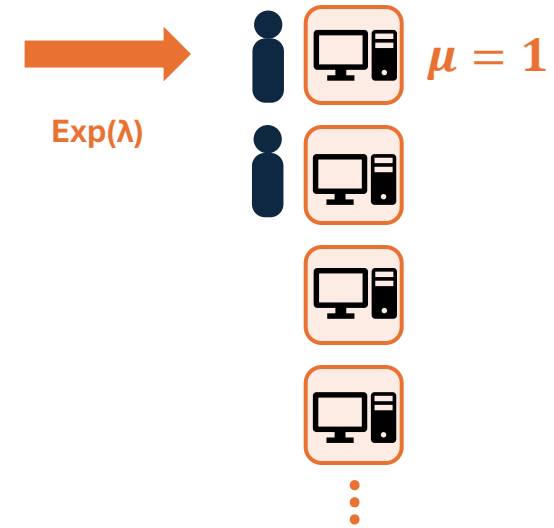

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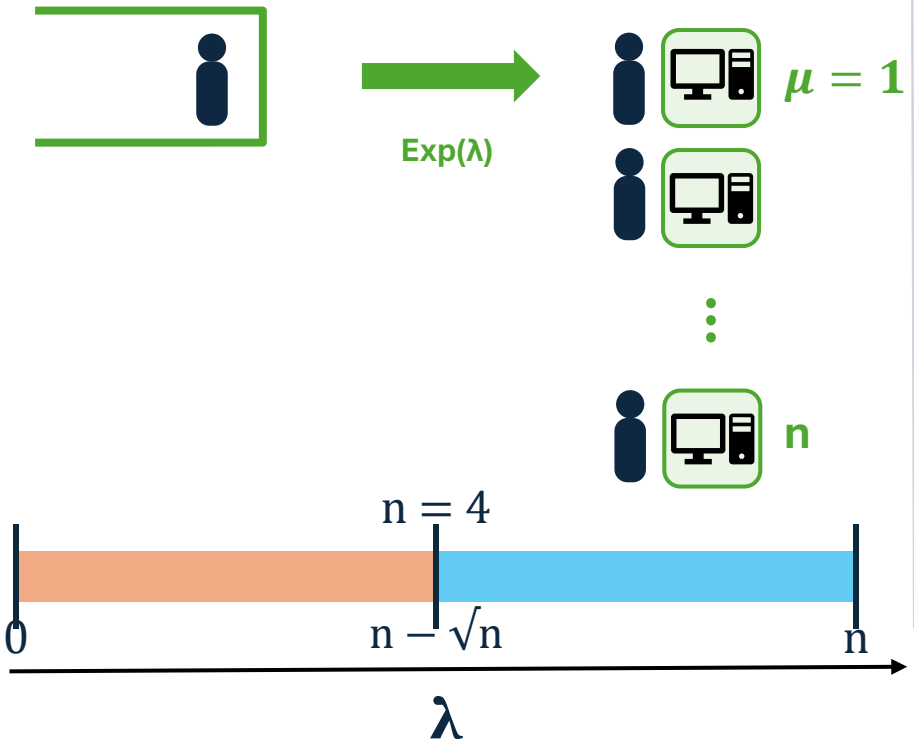

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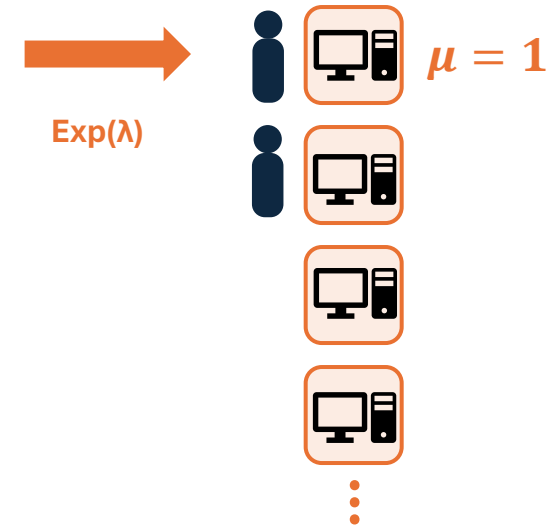

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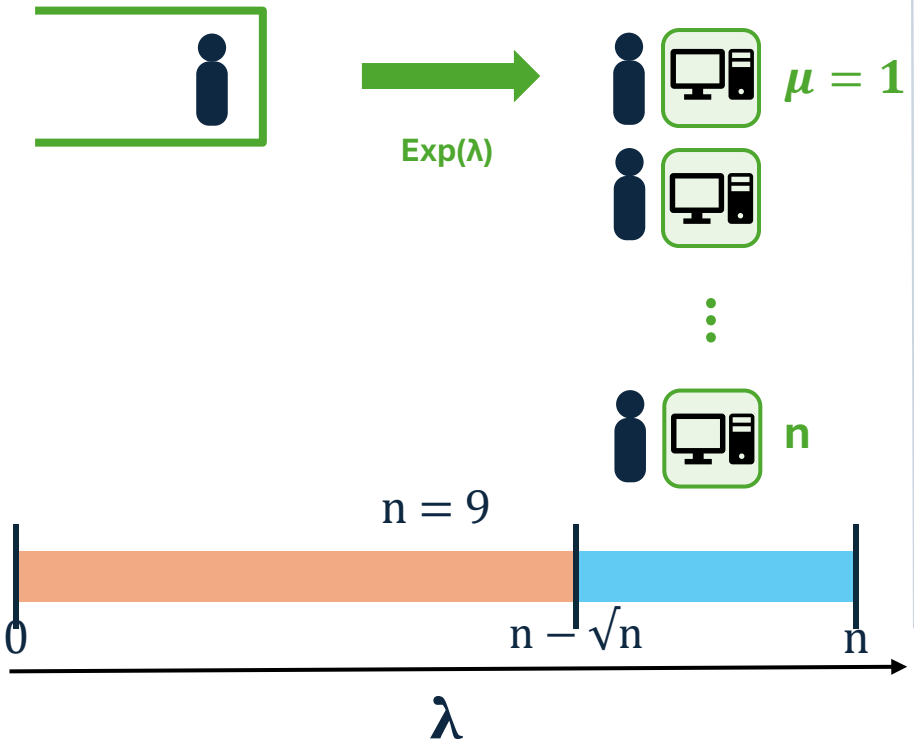

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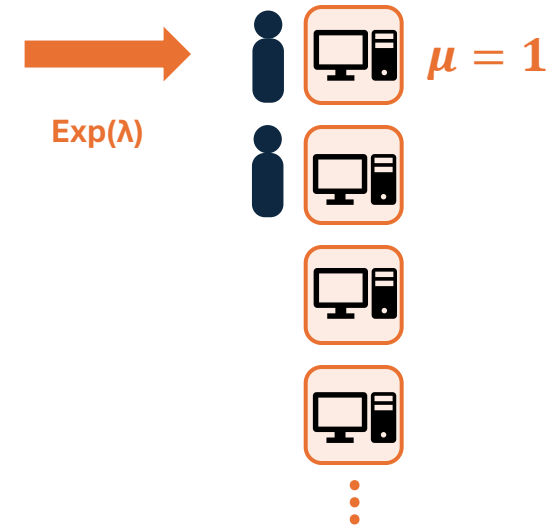

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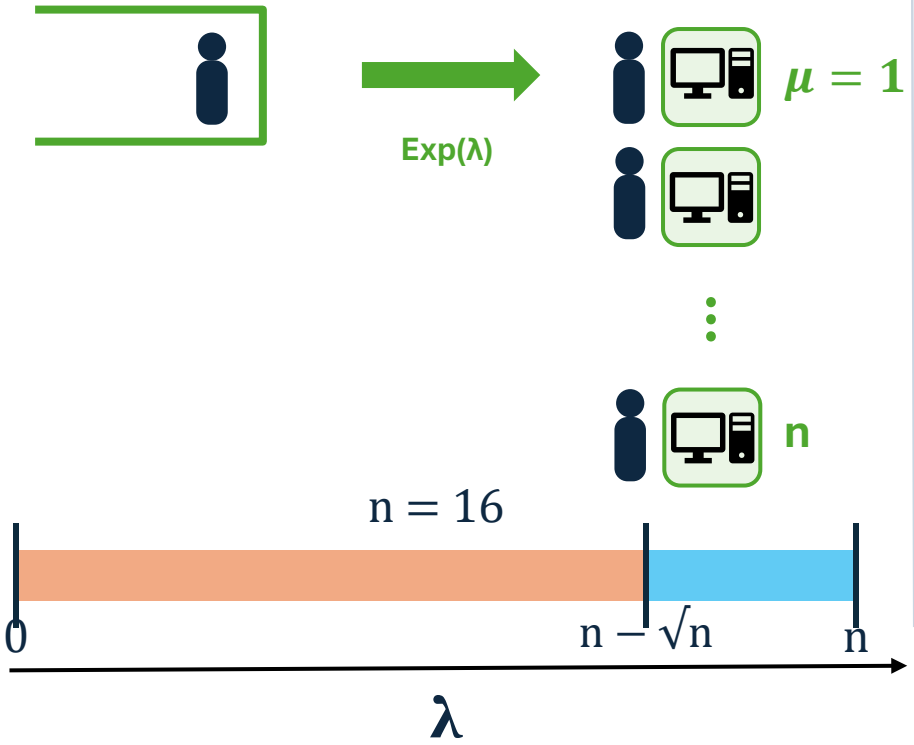

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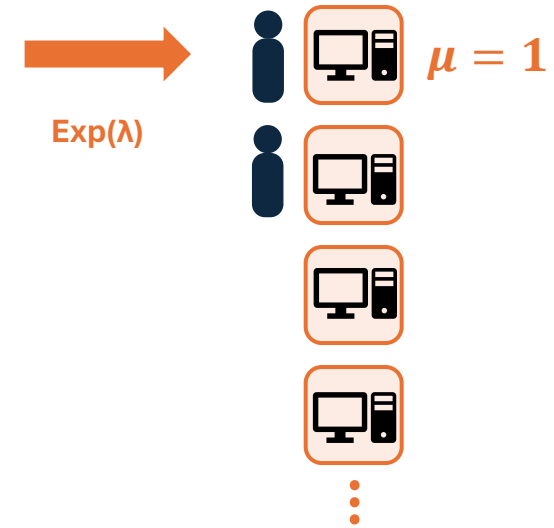

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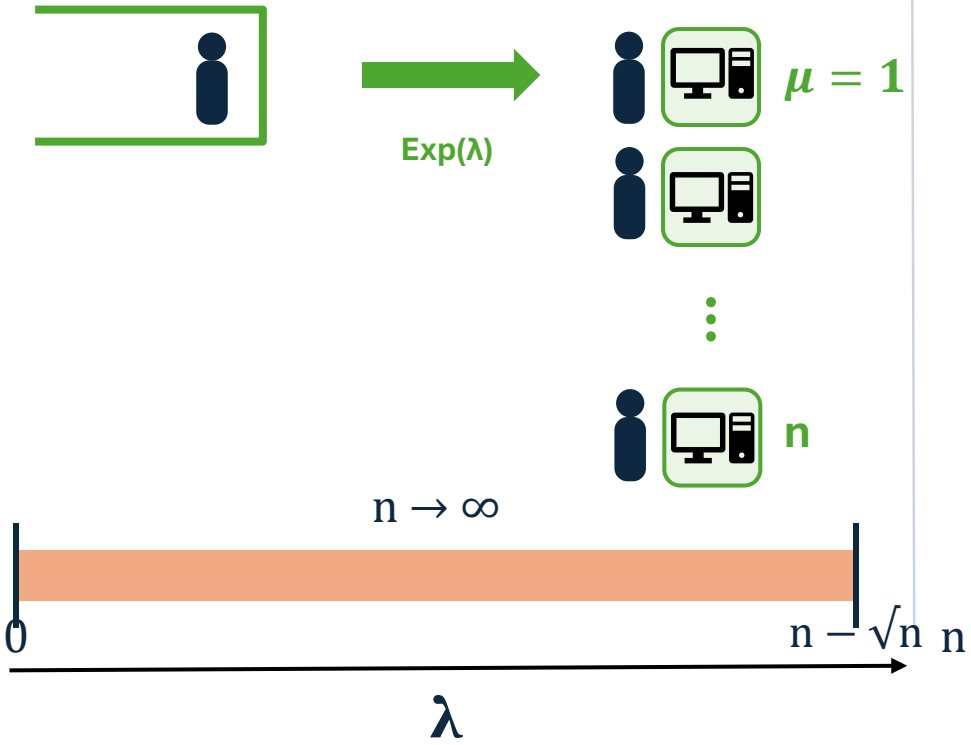

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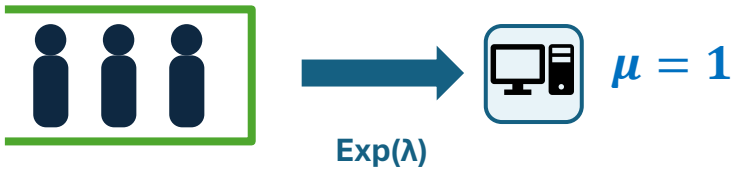

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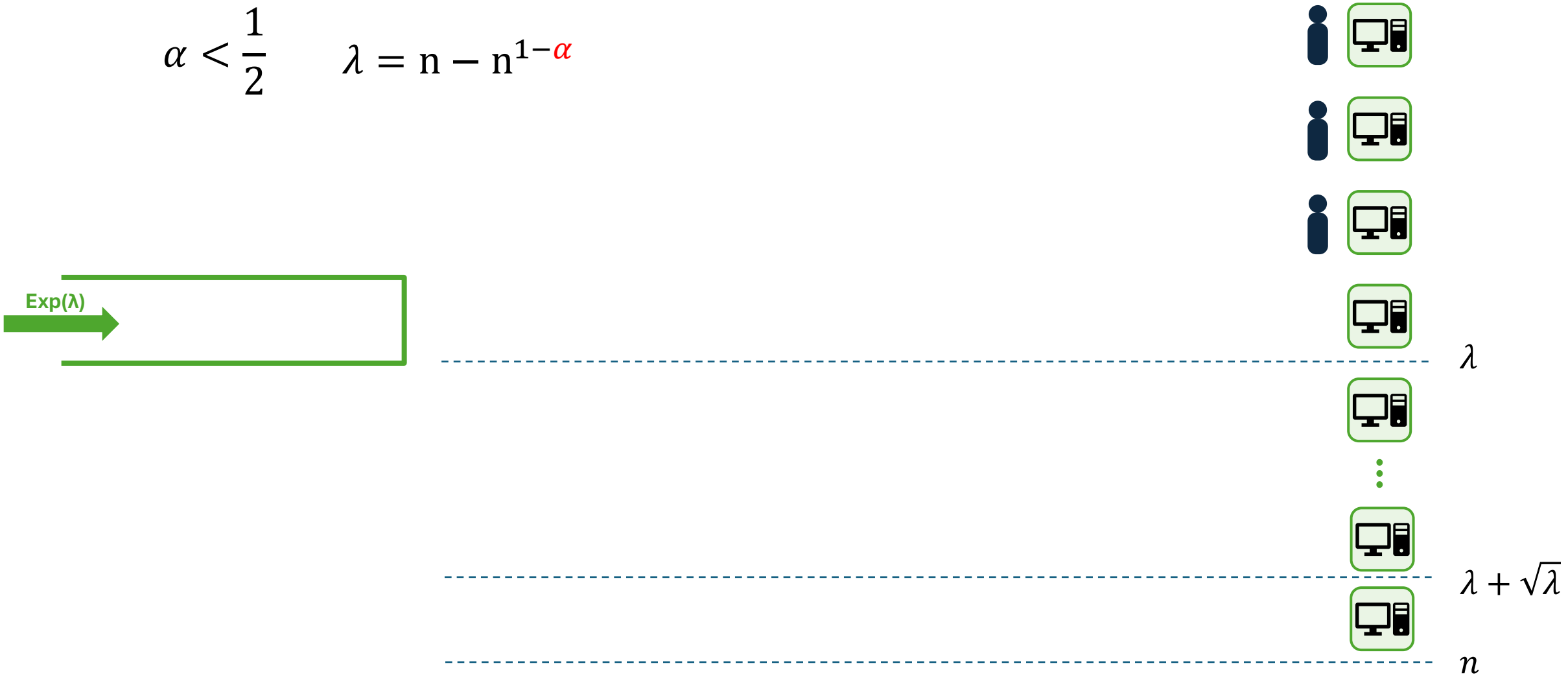
Understanding the phase transition

$$\alpha < \frac{1}{2} \qquad \lambda = n - n^{1-\alpha}$$



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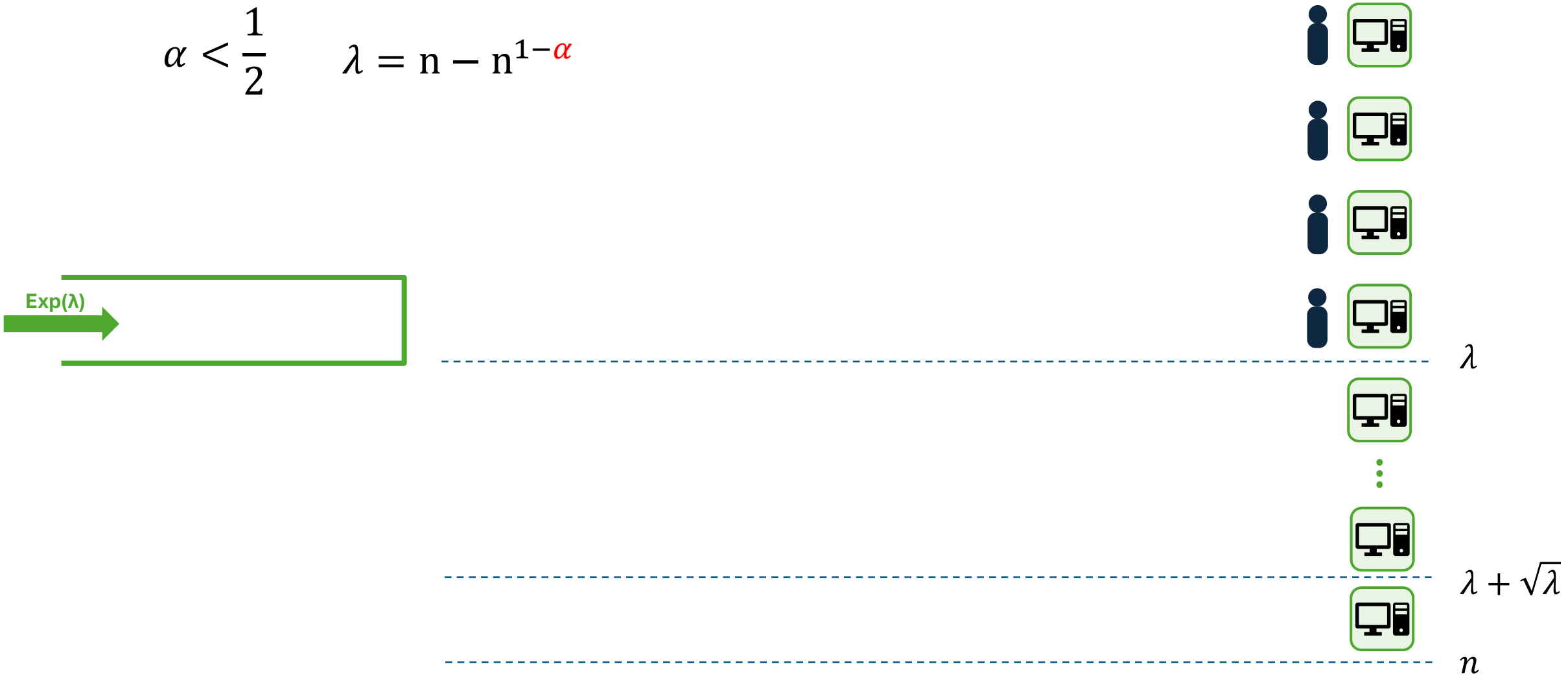

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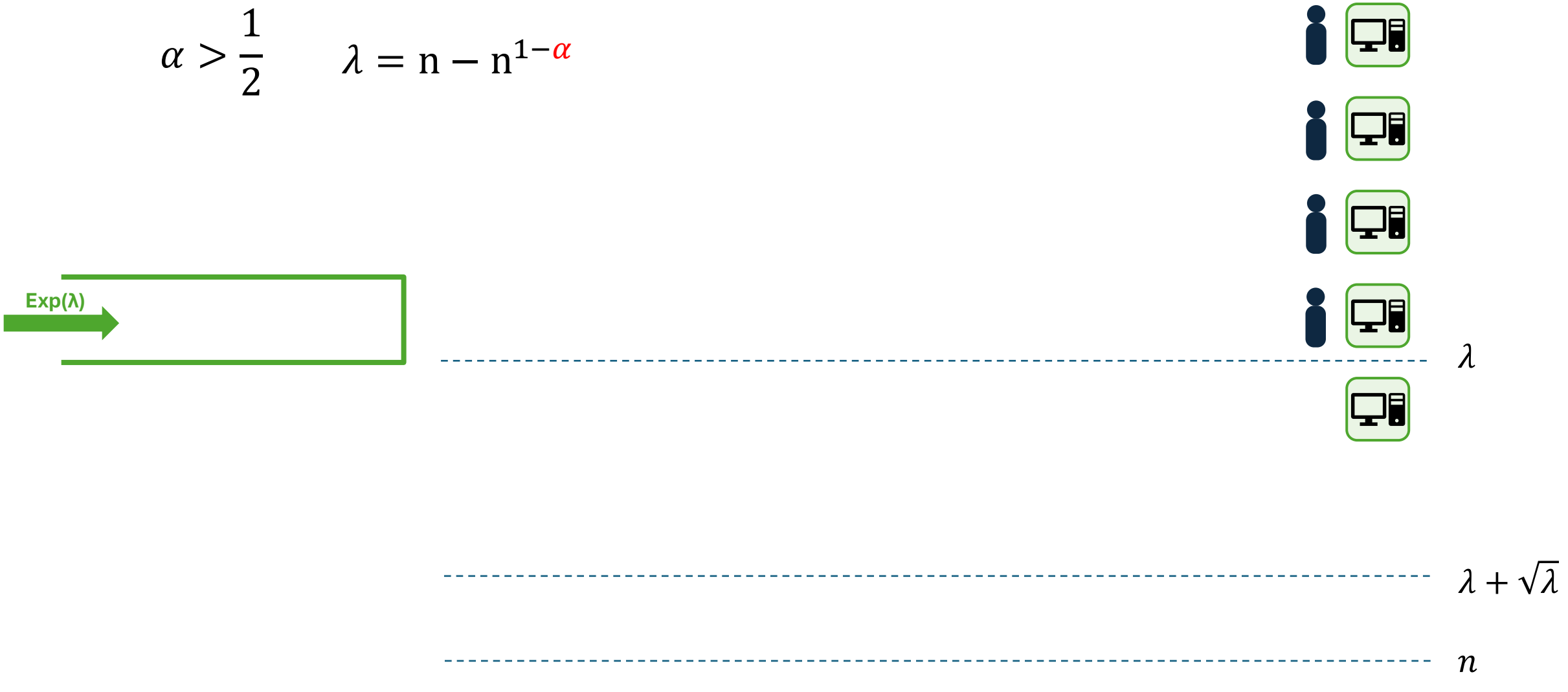

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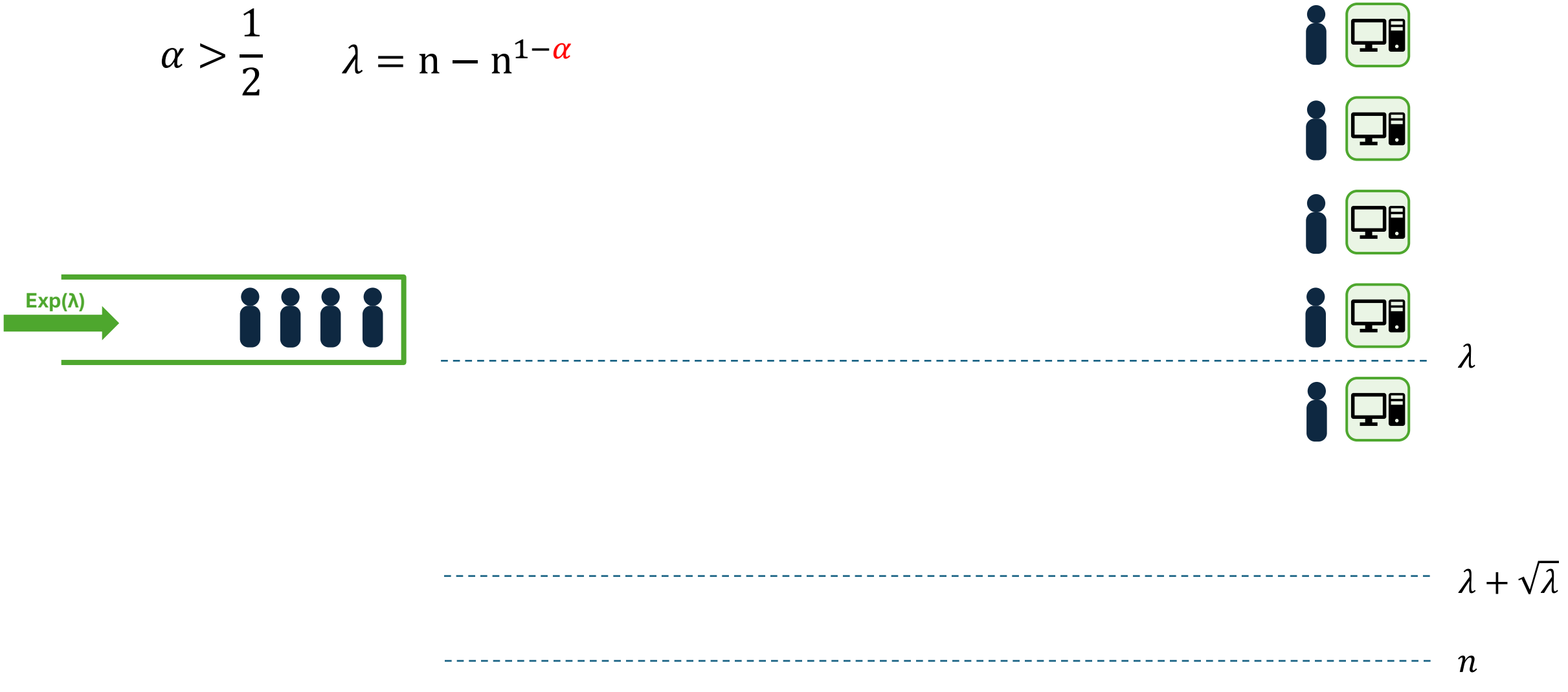

Tightness of the mixing rate

Non-asymptotic rate

Theorem 1 [NVM 2025]

Sub-Halfin-Whitt regime

$$\mathfrak{Rate} = \Omega(1)$$

$n \to \infty$

Super-Halfin-Whitt regime

$$\mathfrak{Rate} = C(\sqrt{n} - \sqrt{\lambda})^2$$

$n \to \infty$

Limiting rate

Theorem 2 [NVM 2025]

$$\mathfrak{Rate} \to 1$$

$$\frac{\mathfrak{Rate}}{(\sqrt{n} - \sqrt{\lambda})^2} \to 1$$

f-divergence

$$D_f(P||Q) = \sum Q(x) f\left(\frac{P(x)}{Q(x)}\right)$$

- f convex
- $f(1) = 0$

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If $f(x) = |x - 1|$ then we have **TV** divergence

f-divergence

$$D_f(P||Q) = \sum Q(x) f\left(\frac{P(x)}{Q(x)}\right)$$

- f convex
- $f(1) = 0$

If $f(x) = x \log x$ then we have **KL** divergence

f-divergence

$$D_f(P||Q) = \sum Q(x) f\left(\frac{P(x)}{Q(x)}\right)$$

- f convex
- $f(1) = 0$

If $f(x) = (x-1)^2$ then we have **Chi-square** divergence

f-divergence

$$D_f(P||Q) = \sum Q(x) f\left(\frac{P(x)}{Q(x)}\right)$$

If $f(x) = (x-1)^2$ then we have **Chi-square** divergence

$$\chi^2(P,Q) = \sum_{x \in \mathcal{S}} \frac{(P(x)-Q(x))^2}{Q(x)}, \qquad \chi(P,Q) = \sqrt{\chi^2(P,Q)}$$

f-divergence

$$D_f(P||Q) = \sum Q(x) f\left(\frac{P(x)}{Q(x)}\right)$$

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Implications

f-divergence

$$D_f(P||Q) = \sum Q(x) f\left(\frac{P(x)}{Q(x)}\right)$$

If $f(x) = (x-1)^2$ then we have **Chi-square** divergence

$$\chi^2(P,Q) = \sum_{x \in \mathcal{S}} \frac{(P(x)-Q(x))^2}{Q(x)}, \qquad \chi(P,Q) = \sqrt{\chi^2(P,Q)}$$

Implications

Other distances

$$d_{\mathrm{TV}}(P,Q) \leq \tfrac{1}{2}\chi(P,Q)$$

$$W_1(P,Q) \leq \chi(P,Q)\sqrt{\mathrm{Var}_Q(X)}$$

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$$\mathbb{P}_P(X \ge x) \le \inf_{\theta > 0} e^{-\theta x}\mathbb{E}_P\left[e^{\theta X}\right]$$

Finite-time Corollaries

Theorem 1 [NVM 2025]

Sub-Halfin-Whitt regime

$$\mathfrak{Rate} = \Omega(1)$$

Super-Halfin-Whitt regime

$$\mathfrak{Rate} = C(\sqrt{n} - \sqrt{\lambda})^2$$

Mean queue length bound

$$\mathbb{E}[q(t)] \leq \underbrace{\mathbb{E}[q(\infty)]}_{\lambda + O(n^{0.5 \vee \alpha})} + e^{-\mathfrak{Rate} \times t} \chi\left(q(0), q(\infty)\right) O\left(n^{0.5 \vee \alpha}\right)$$

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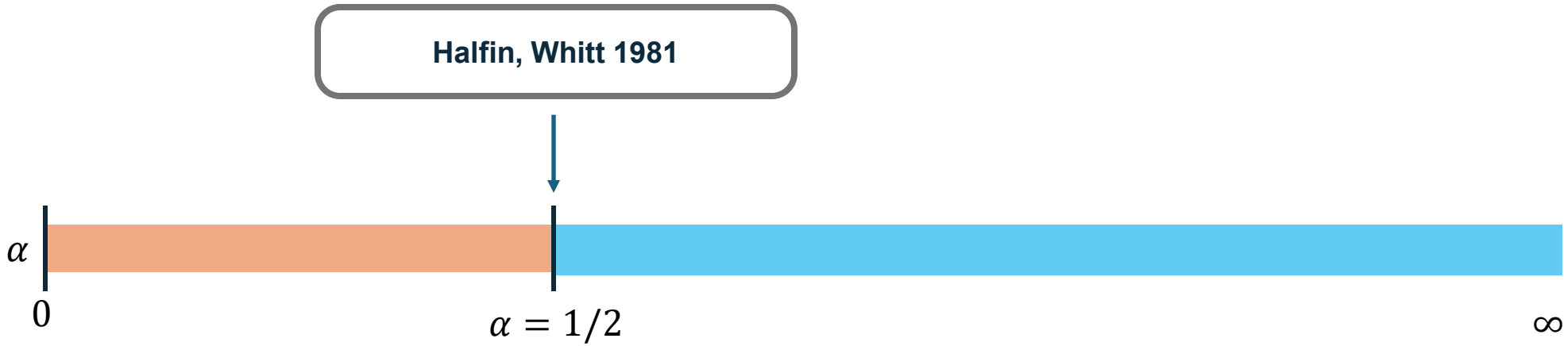
Finite-time tail bound

$$\mathbb{P}\left(q(t) > n + x\right) \leq \left(1 + e^{-\mathfrak{Rate} \times t} \chi\left(q(0), q(\infty)\right)\right) O\left(\sqrt{\frac{x}{1-\rho}}\right) e^{-\frac{x}{2(1-\rho)}}$$

Related works

Steady-state results



Load $\rho = \frac{\lambda}{\mathrm{n}}$

$$\alpha = \frac{\log\left(\frac{1}{1-\rho}\right)}{\log \mathrm{n}}$$

Related works

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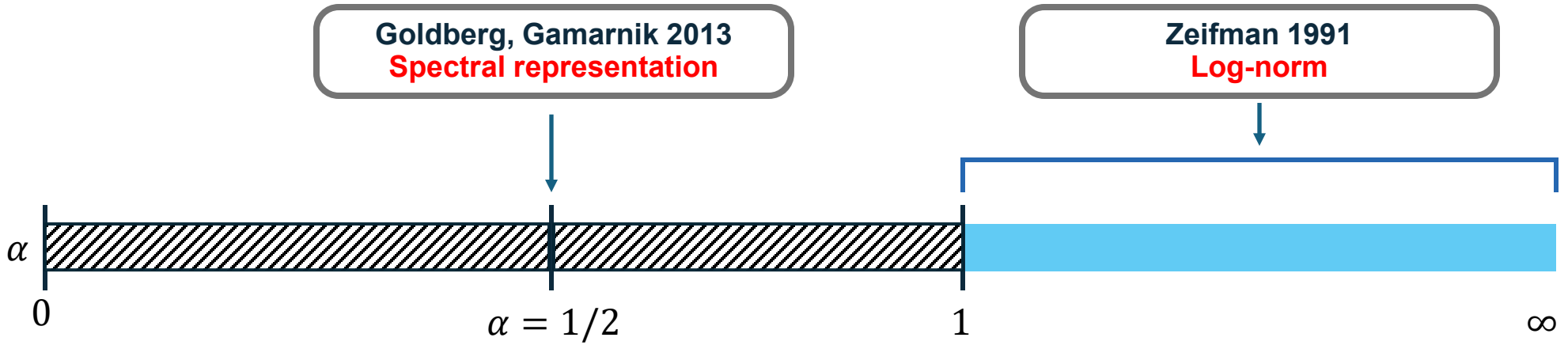

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Related works

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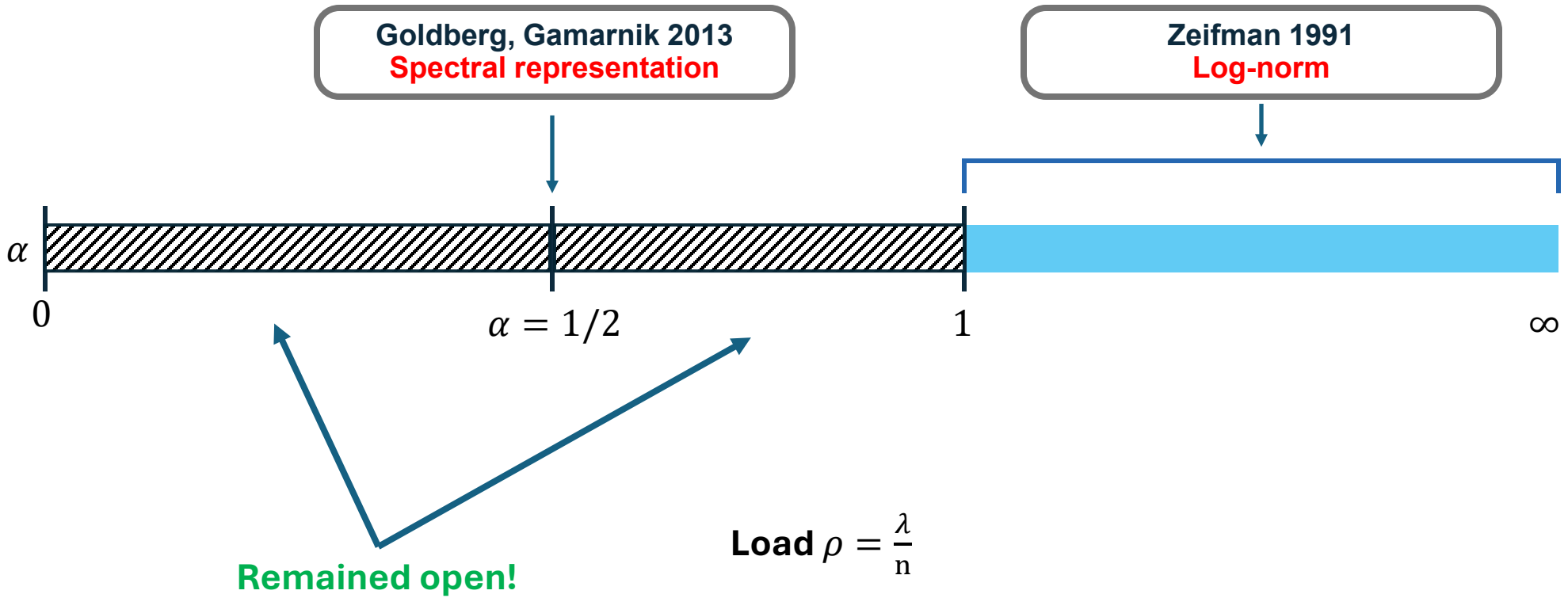

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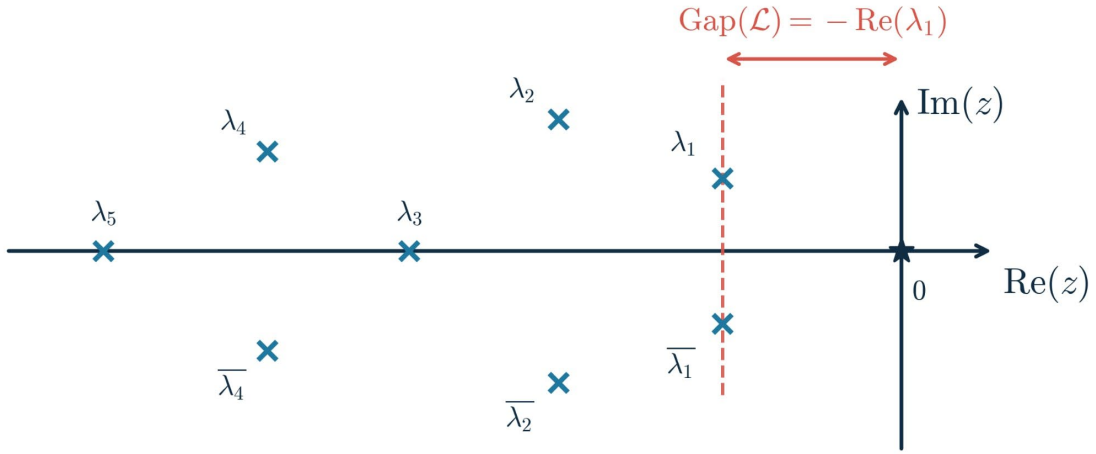
Proof Technique

Mixing time of Markov chains

Finite state CTMC

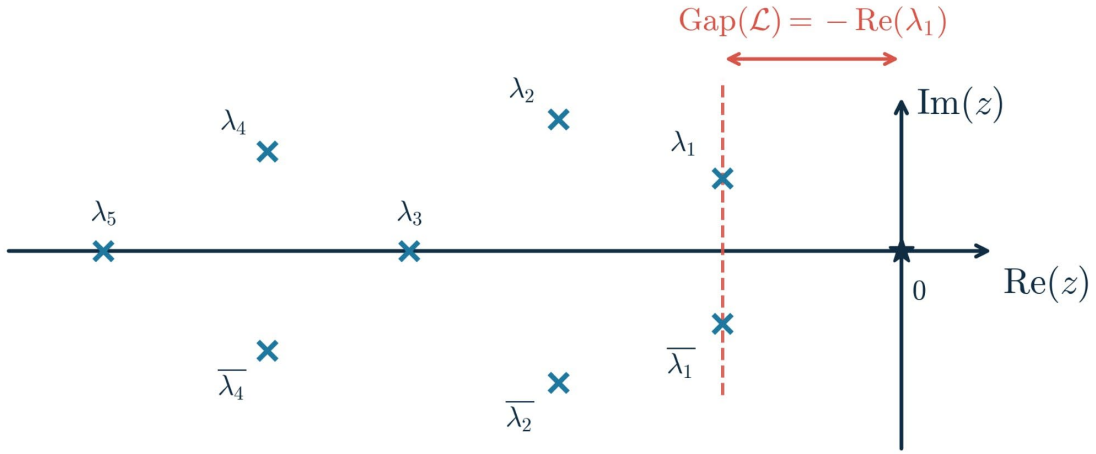


Eigenvalues of $\mathcal{L}$

$$\chi(p(t), \pi) \leq e^{-\mathrm{Gap}(\mathcal{L}) \times t} \chi(p(0), \pi)$$

Mixing time of Markov chains

Finite state Reversible CTMC



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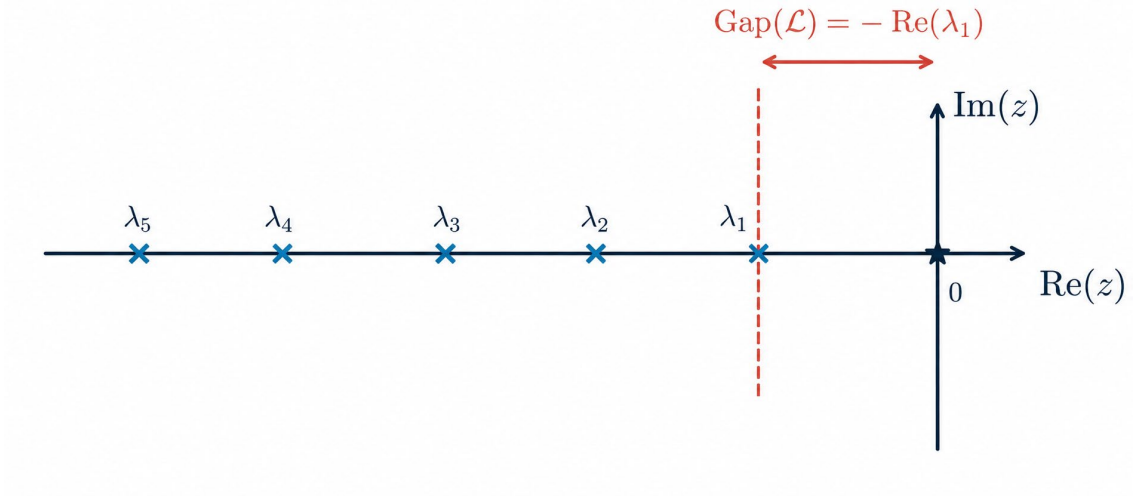

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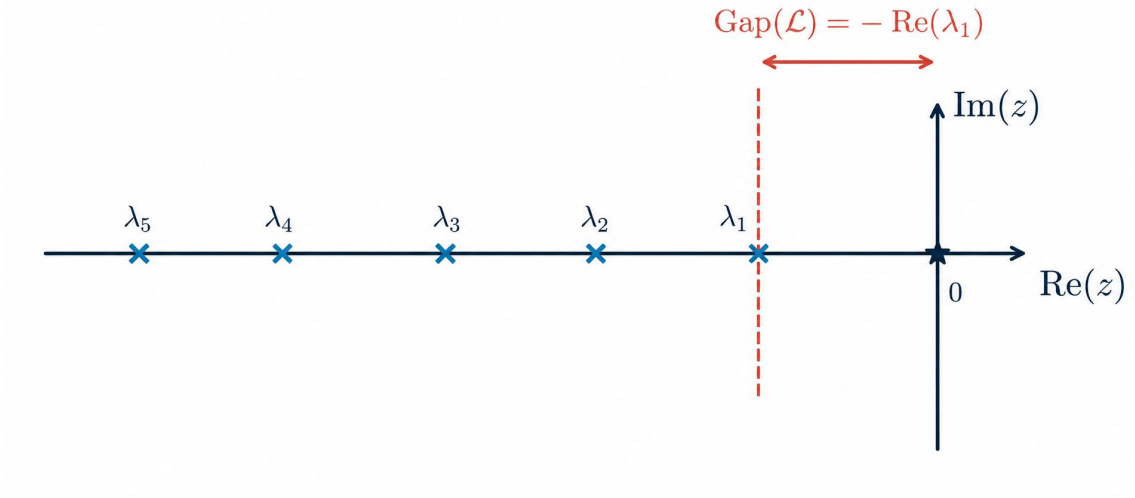

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Courant-Fischer Minimax Theorem

Mixing time of Markov chains

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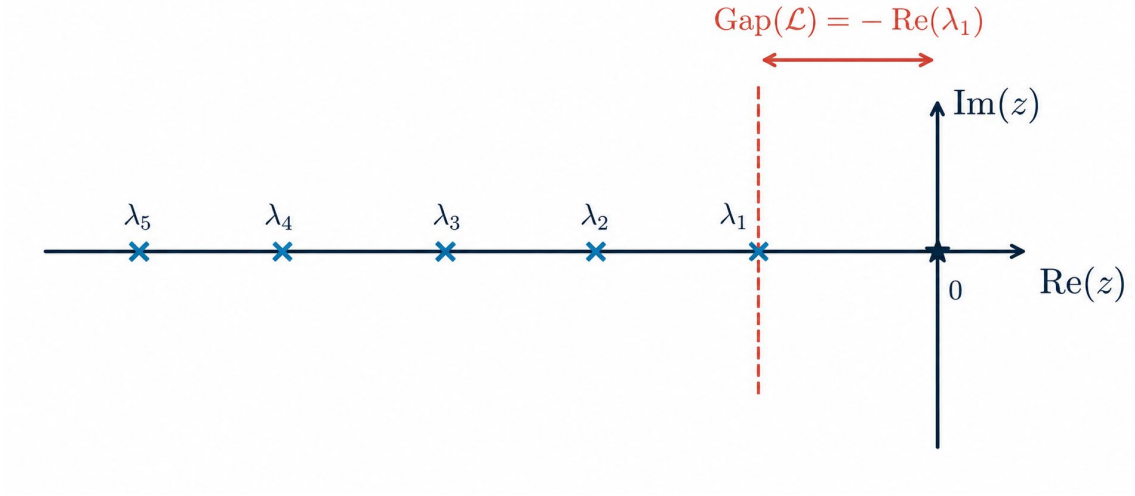

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Mixing time of Markov chains

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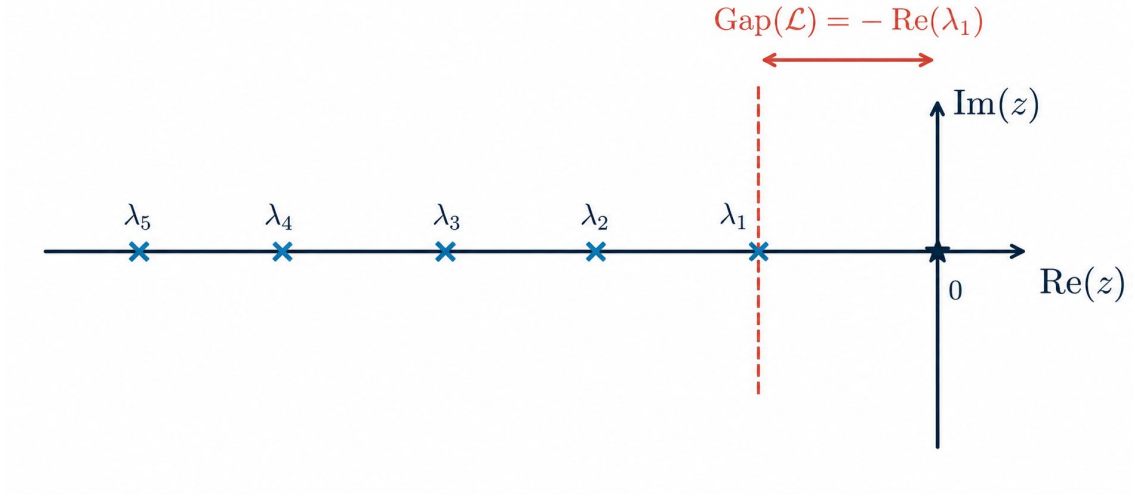

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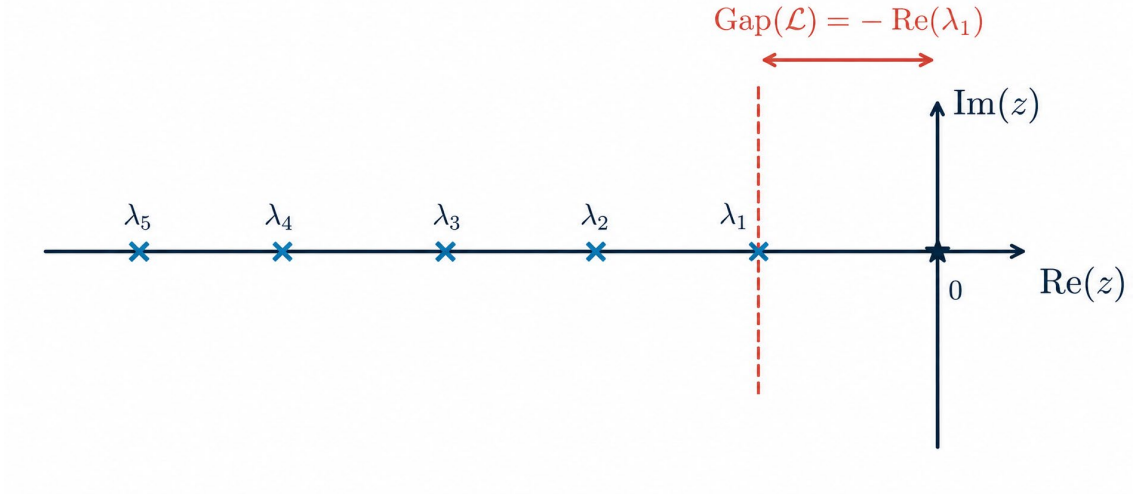

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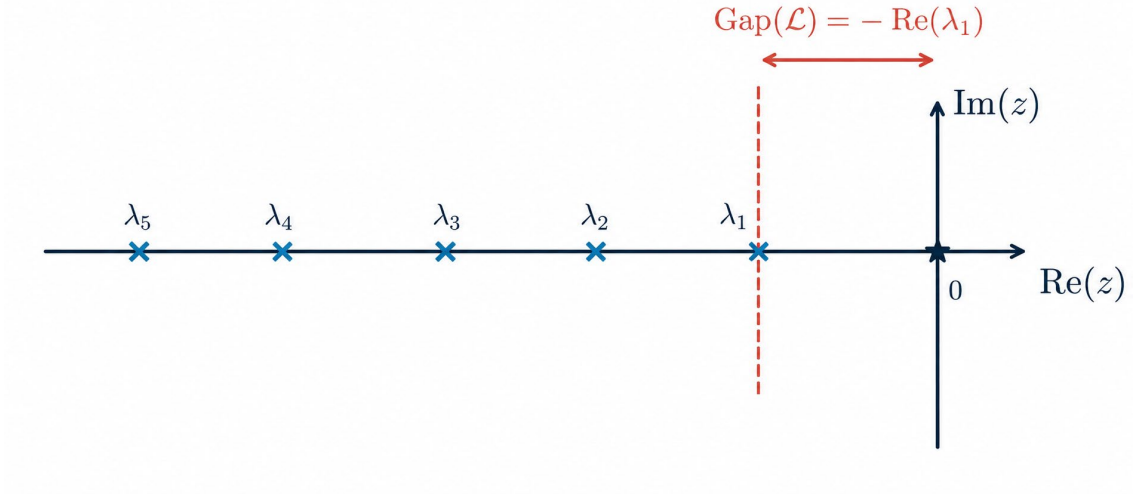
Goal: Find Poincaré constant

Mixing time of Markov chains

Finite state Reversible CTMC



Eigenvalues of $\mathcal{L}$

$$\mathrm{Gap}(\mathcal{L}) = \inf_{\substack{f \in \ell_{2,\pi}: \\ \mathrm{Var}_\pi(f) > 0}} \frac{\langle f, -\mathcal{L}f \rangle_\pi}{\mathrm{Var}_\pi(f)}$$

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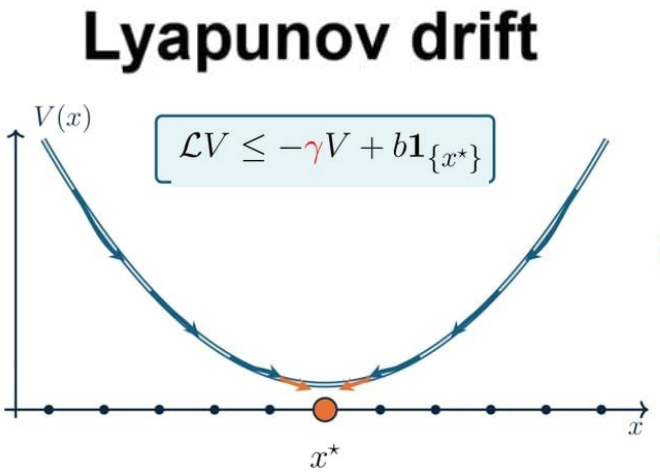
Goal: Find Poincaré constant

> *Algebra is the offer made by the devil to the mathematician. The devil says: "I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine."*
>
> Michael Atiyah

Lyapunov drift approach



> *Of course we like to have things both ways; we would probably cheat on the devil, pretend we are selling our soul, and not give it away.*
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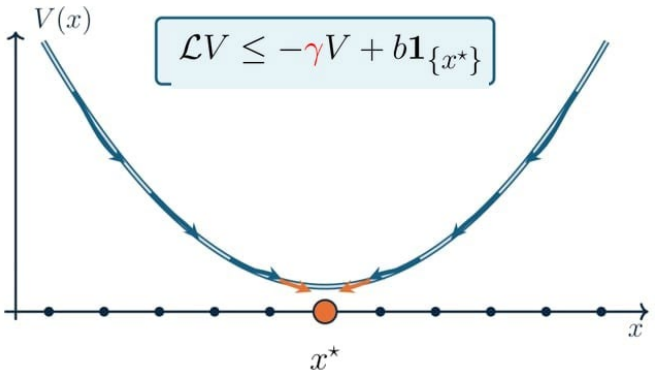
Lyapunov drift approach

Lyapunov drift



Poincaré inequality

$$\mathrm{Var}_\pi(f) \leq \frac{1}{\gamma} \langle f, -\mathcal{L}f \rangle_\pi$$

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Lyapunov drift approach

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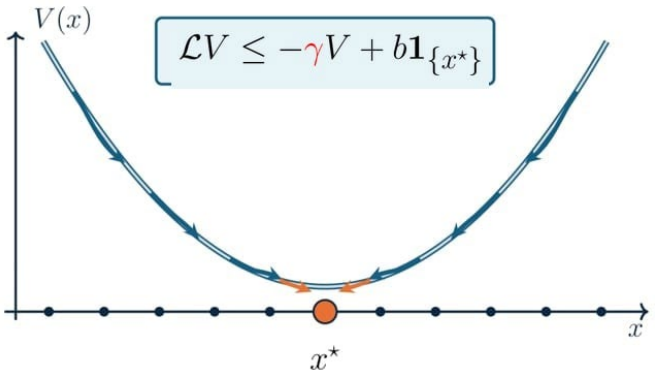

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Spectral gap

$$\mathfrak{Rate} \geq \gamma$$

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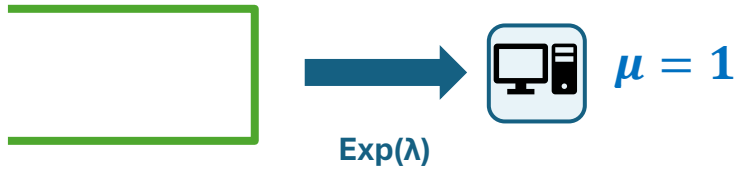
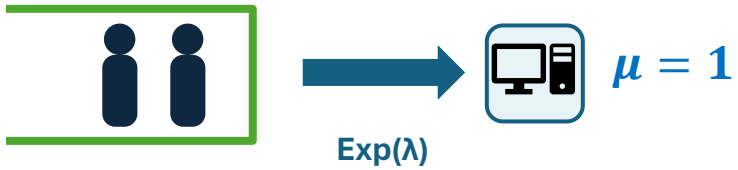
Example: M/M/1



$$\dot{V} \approx -\epsilon\theta V$$

Drift: $\lambda > 0$

Example: M/M/1



$$\dot{V} \approx -\epsilon\theta V$$

Drift: $\lambda - 1 = -\epsilon$

Example: M/M/1



ODE $\dot{q} = -\epsilon$ $\qquad \dot{V} \approx -\epsilon\theta V$ $\qquad$ **Drift:** $\lambda - 1 = -\epsilon$

Example: M/M/1



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$\dot{V} \approx -\epsilon\theta V$

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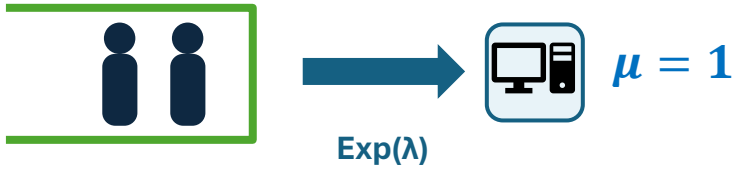
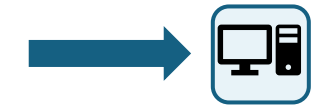


ODE $\dot{q} = -\epsilon$ **+** $V(q) = e^{\theta q}$ $\dot{V} \approx -\epsilon\theta V$ **Drift:** $\lambda - 1 = -\epsilon$

Example: M/M/1





Exp(λ)

$\mu = 1$



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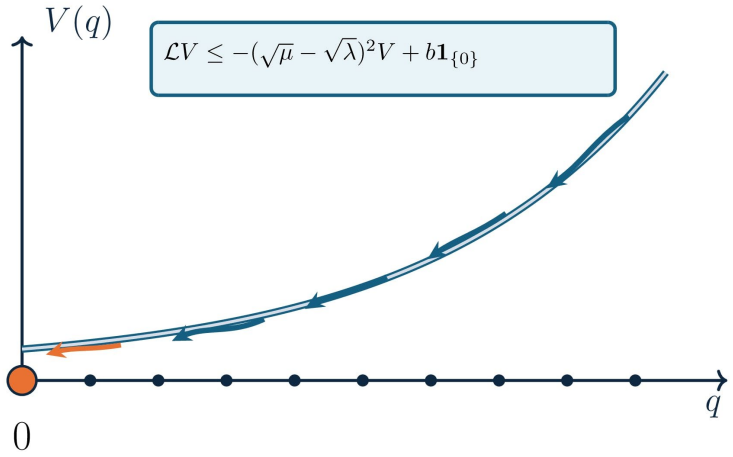
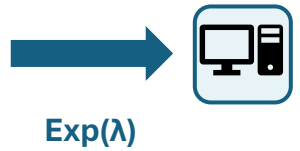


Example: M/M/1





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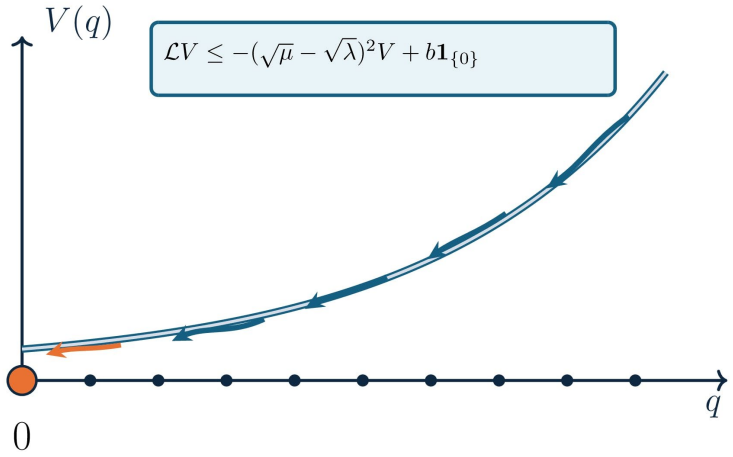

Poincaré constant $C_P = \frac{1}{(\sqrt{\mu} - \sqrt{\lambda})^2}$

Example: M/M/1





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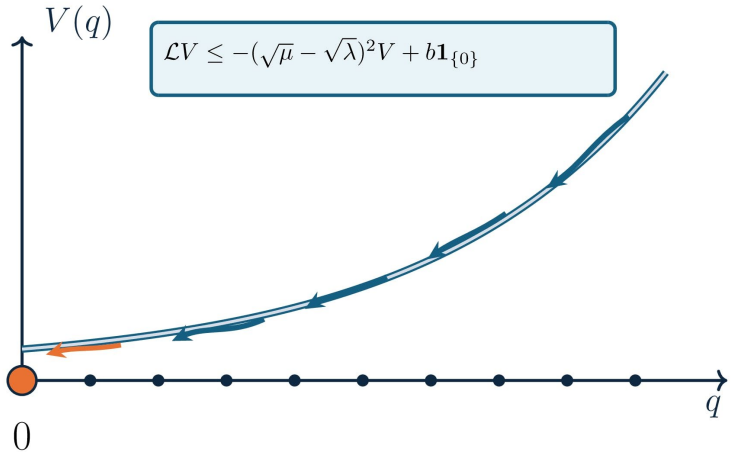

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Mixing bound $\mathfrak{Rate} \geq (\sqrt{\mu} - \sqrt{\lambda})^2$

Example: M/M/1





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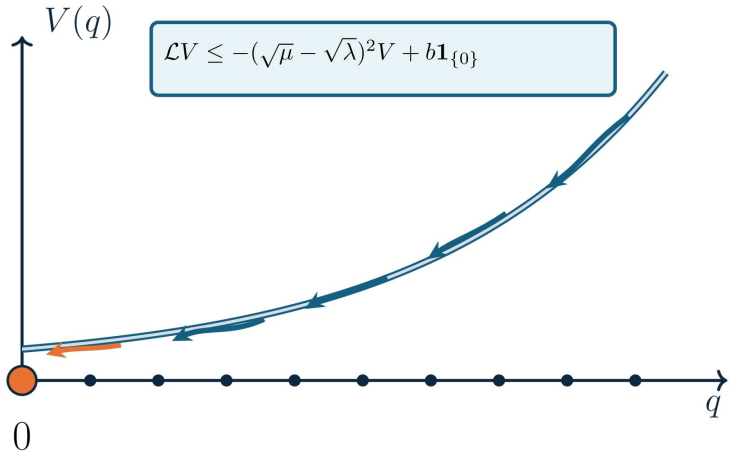

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Mixing bound $\mathfrak{Rate} \geq (\sqrt{\mu} - \sqrt{\lambda})^2$

Matches the literature!

Mixing results: All regimes

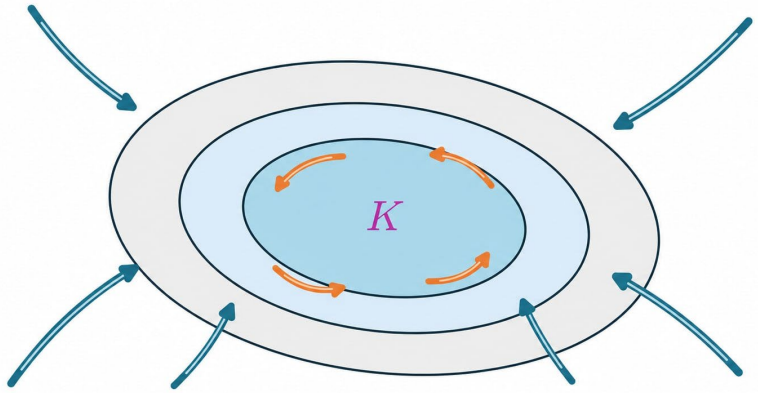
$$\chi(q(t), q(\infty)) \leq e^{-\mathfrak{Rate} \times t} \chi(q(0), q(\infty))$$

System	$M/M/\infty$	$\alpha \in (0, 1/2)$	$\alpha \in (1/2, \infty)$	$M/M/1$
$\mathfrak{Rate}$	1	$\Omega(1)$	$C(\sqrt{n} - \sqrt{\lambda})^2$	$(\sqrt{\mu} - \sqrt{\lambda})^2$
Lyapunov function				**Exp**

Lyapunov-Poincaré approach

Step 1: Lyapunov drift

$$\mathcal{L}V \leq -\gamma V + b\, 1_K$$



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$$\mathrm{Var}_{\pi_K}(f) \leq \frac{C_L}{\pi(K)} \langle f, -\mathcal{L}f \rangle_\pi$$

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[NVM 2025]

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$$\mathrm{Var}_{\pi_K}(f) \leq \frac{C_L}{\pi(K)}\langle f, -\mathcal{L}f\rangle_\pi$$

Poincaré inequality

$$\mathrm{Var}_\pi(f) \leq C_P\langle f, -\mathcal{L}f\rangle_\pi$$

Step 3: $C_P = \dfrac{1 + bC_L}{\gamma}$

[NVM 2025]

Exponential Mixing

$$\chi(q(t), q(\infty)) \leq e^{-\frac{t}{C_P}}\chi(q(0), q(\infty))$$

Lyapunov-Poincaré approach

Step 1: Lyapunov drift

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Minorization: Rosenthal et al. (1995)
Atom set: Meyn et al. (1996)
Canonical path method: **NVM** (2025)
Truncation method: **NVM** (2025)

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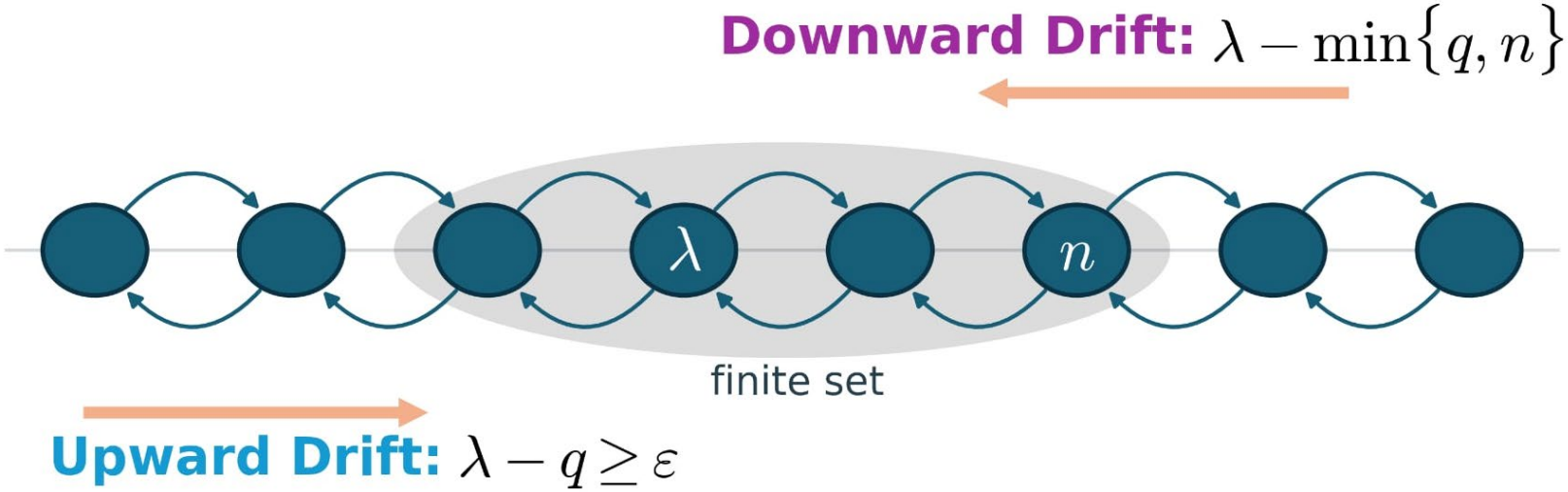

Strong Two-sided drift!

$$\mathcal{L}V \leq -\left(\sqrt{n} - \sqrt{\lambda}\right)^2 V + b1_K \qquad b = O(1)$$

$$\dot{V} \approx -\epsilon\theta V$$

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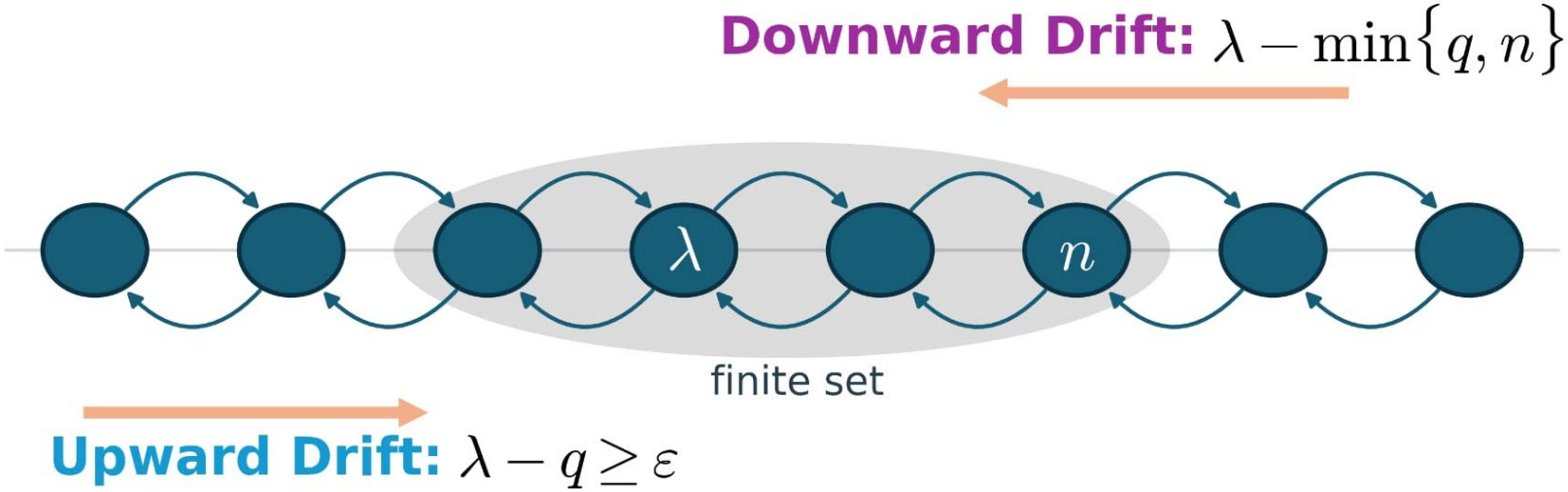

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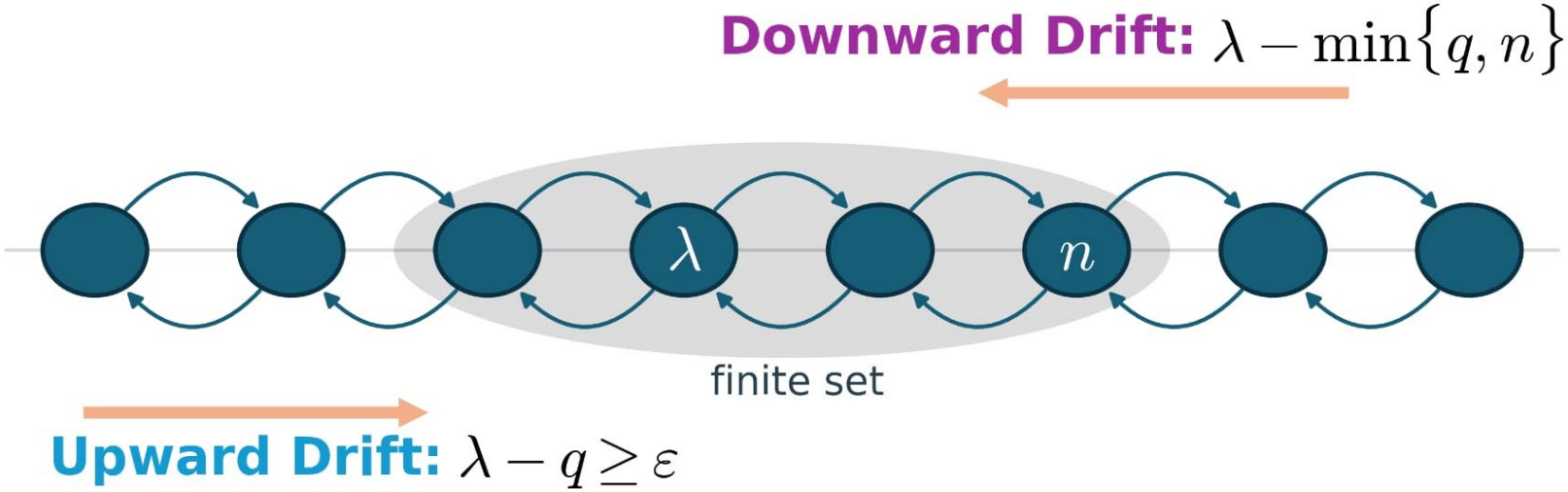

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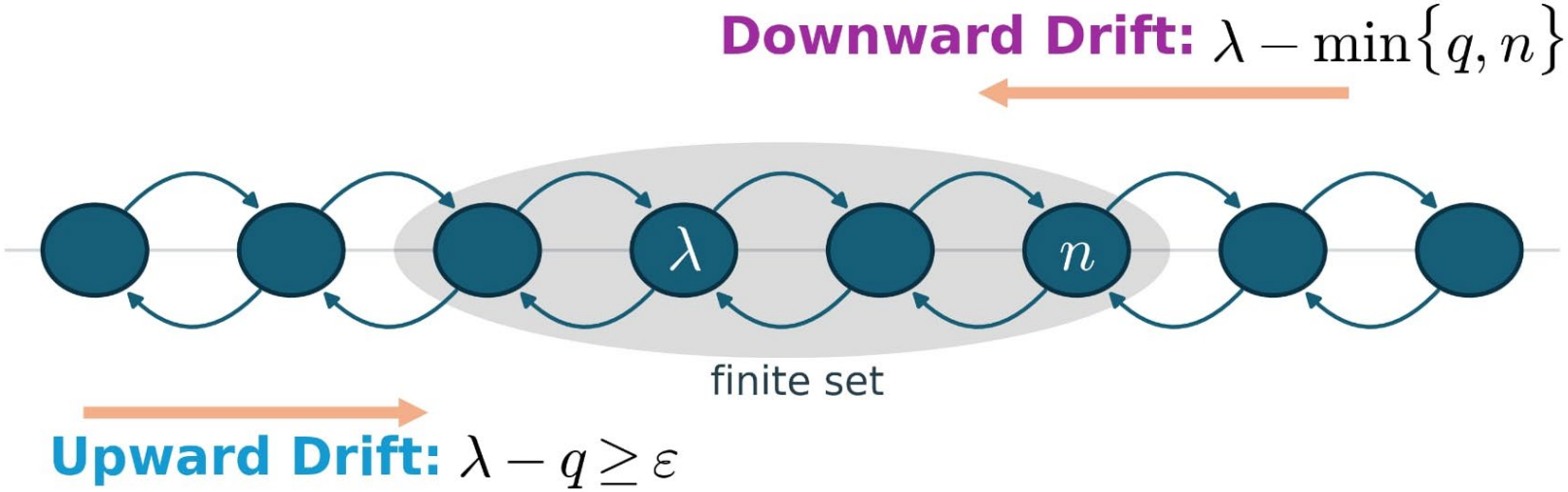

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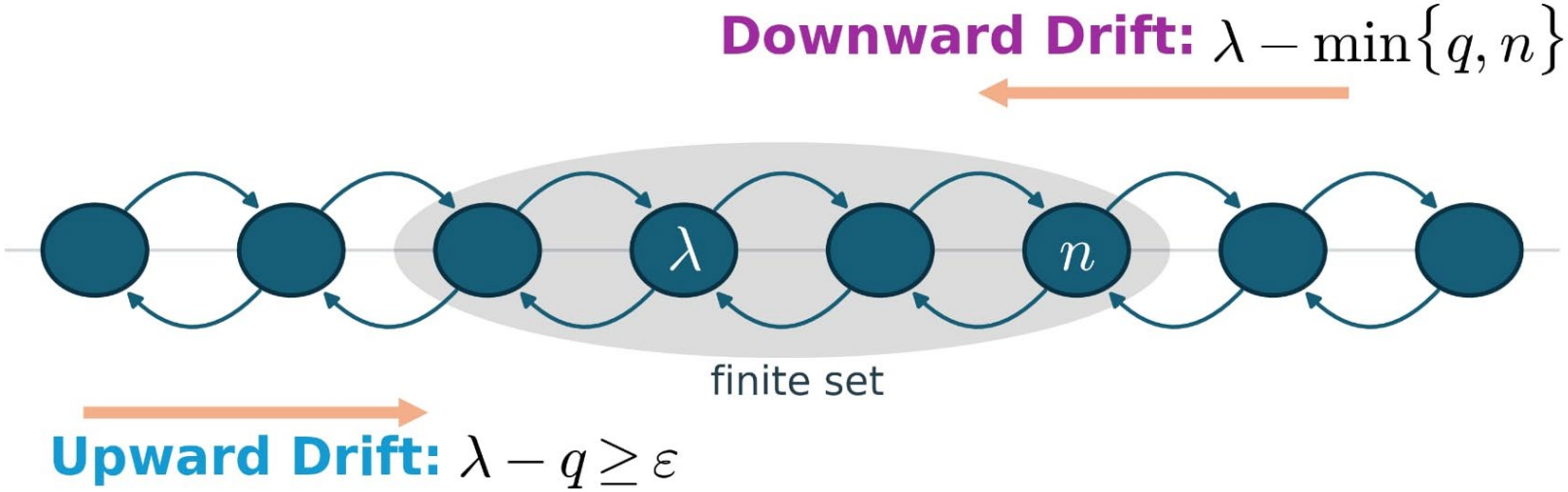

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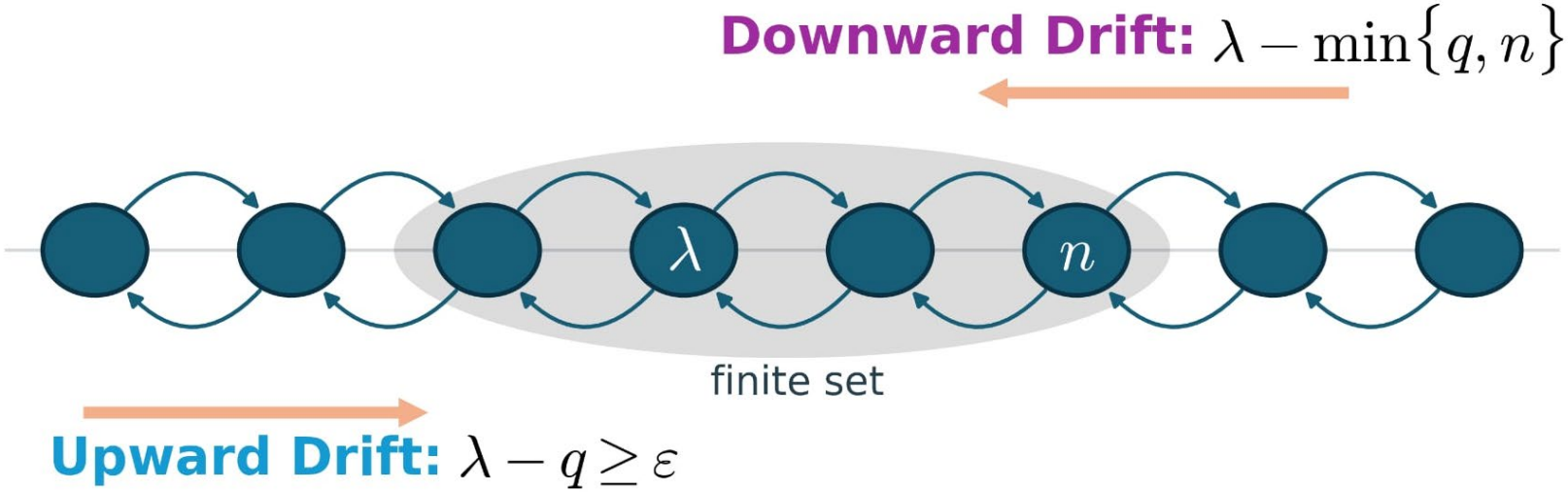

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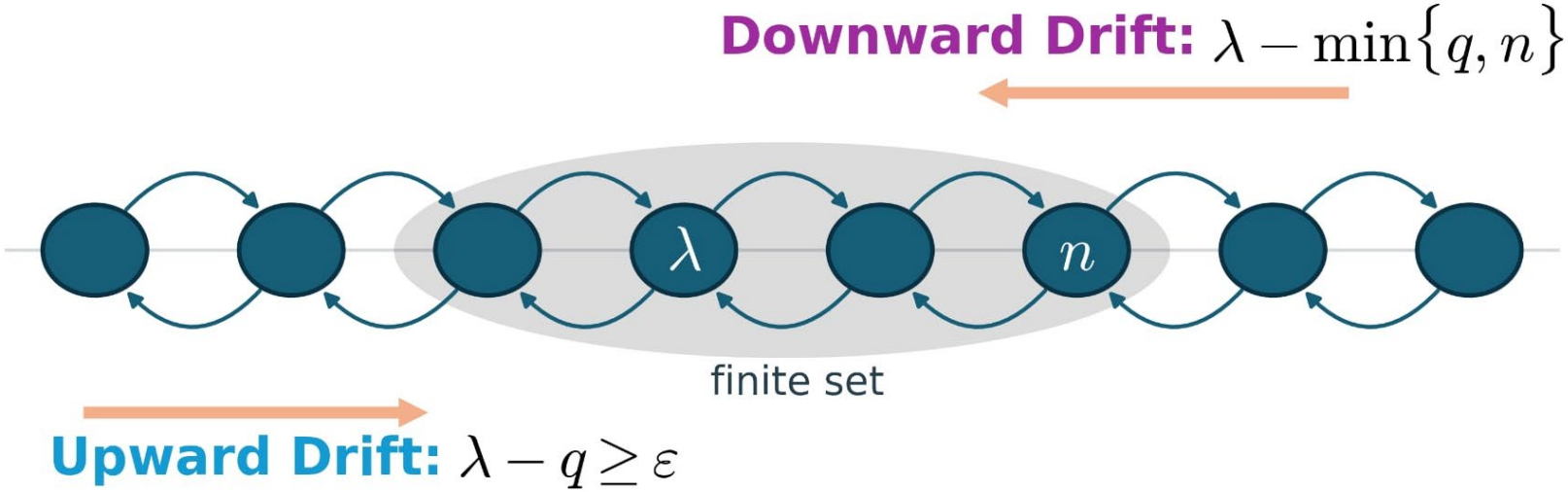

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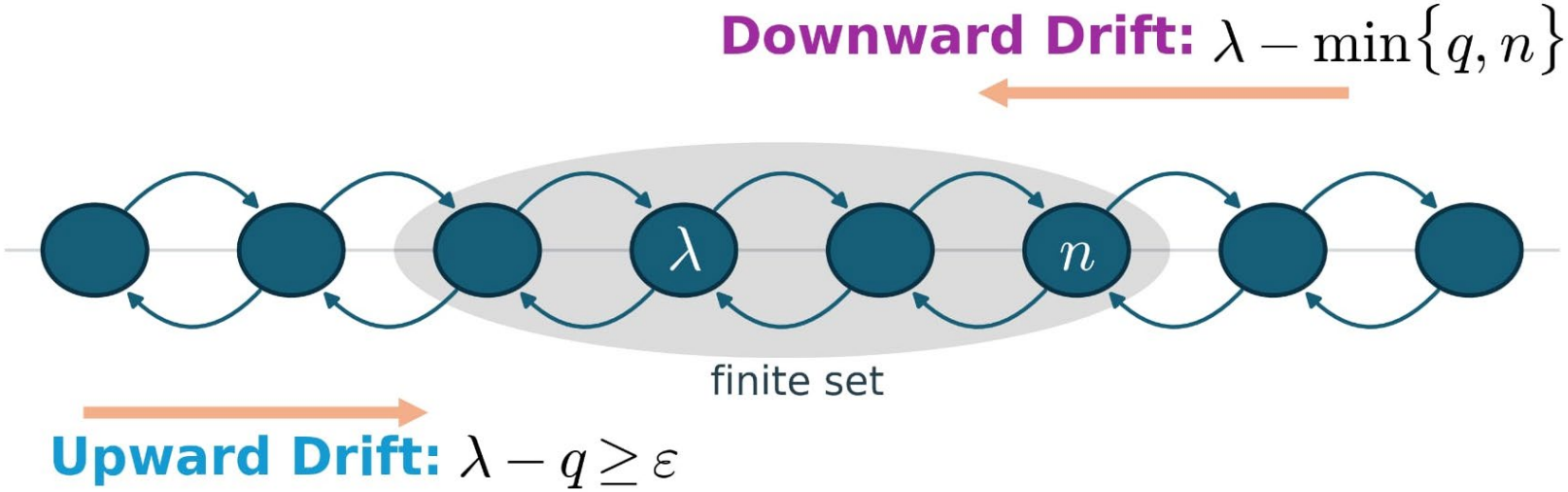

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Idea: Choose $V(q) = e^{\theta[q-n]^+ + \theta[n-q]^+}$ $\quad [x]^+ = \max\{x, 0\}$

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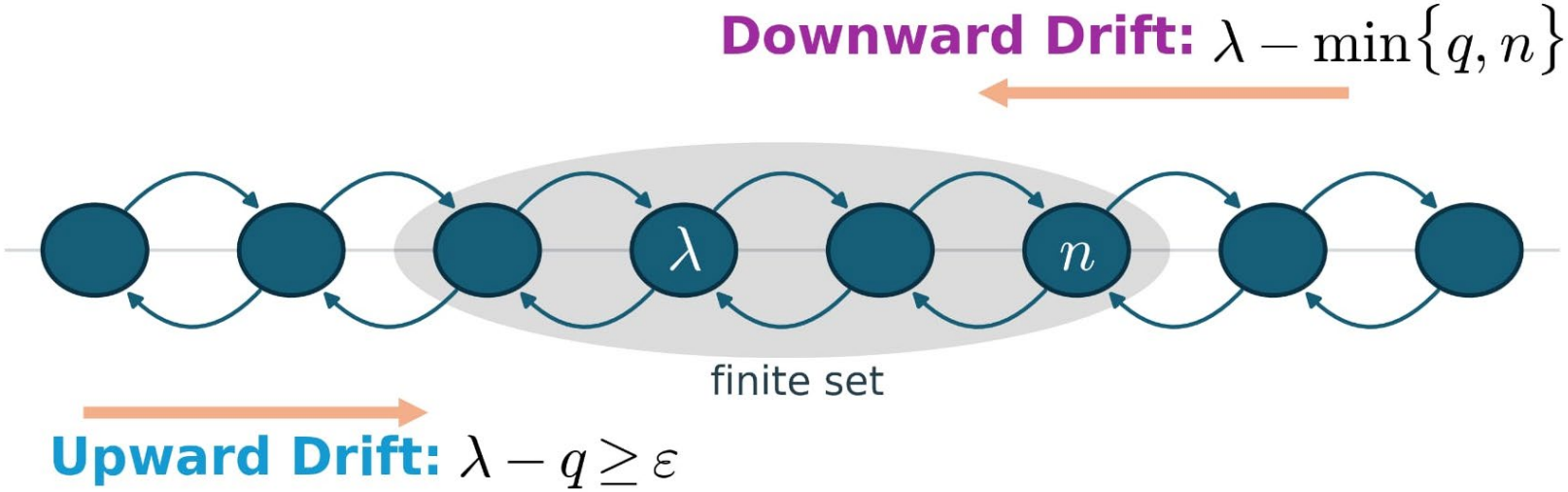

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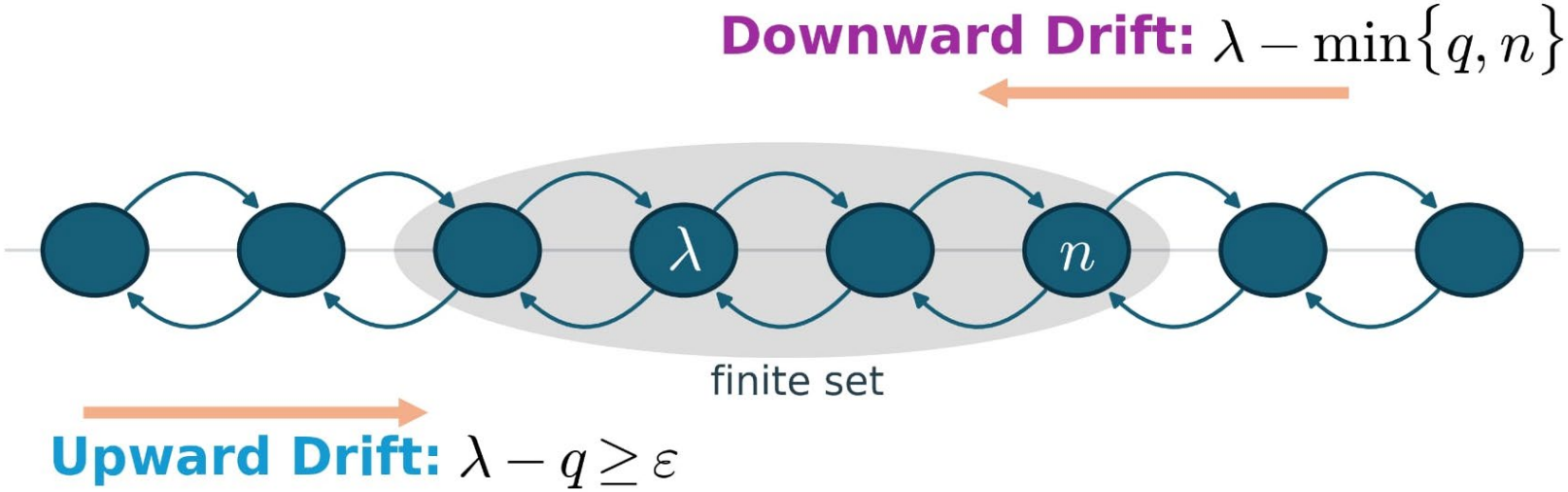

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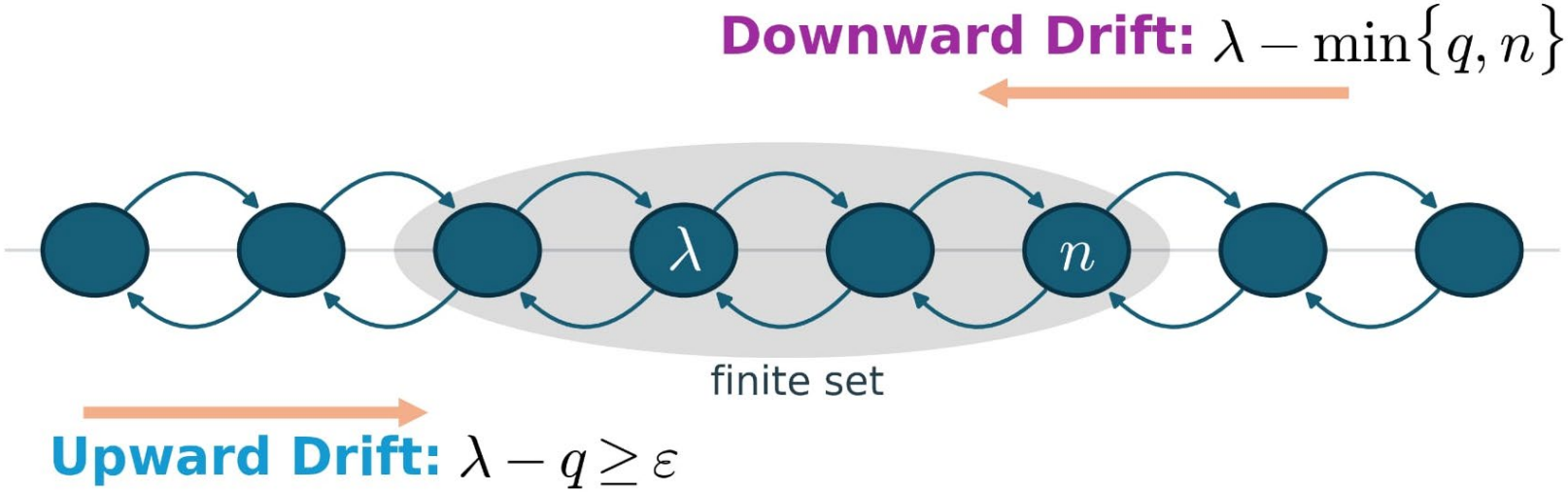

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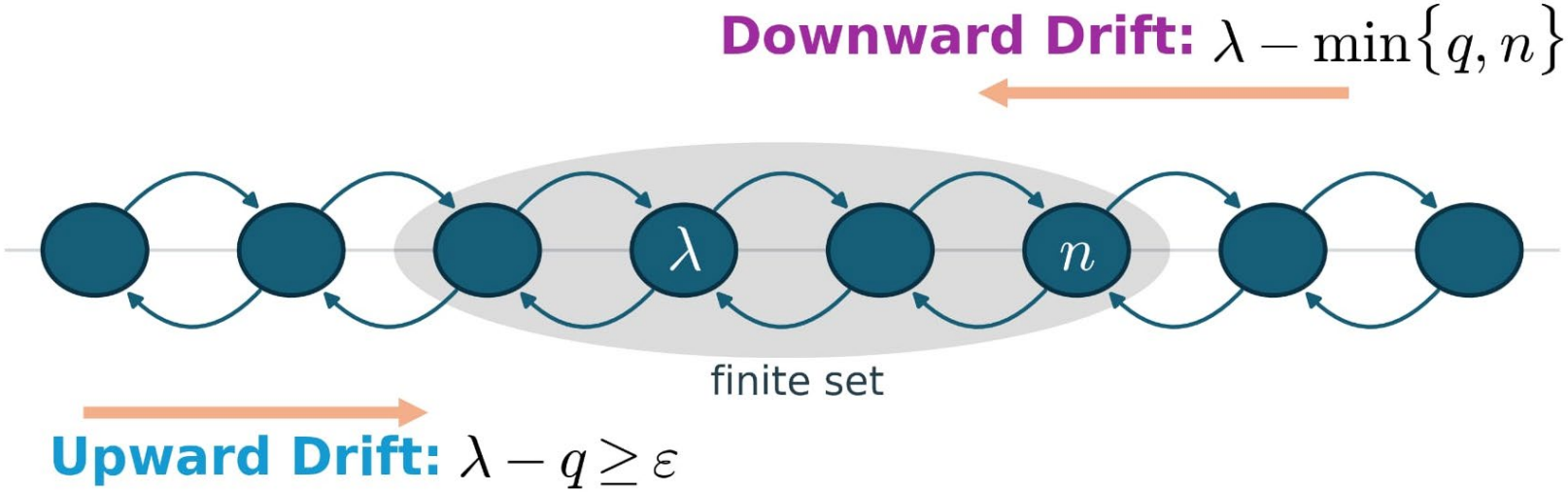

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M/M/1
$e^{-\left(\sqrt{\mu}-\sqrt{\lambda}\right)^2 t}$

Proof framework: Super-HW regime

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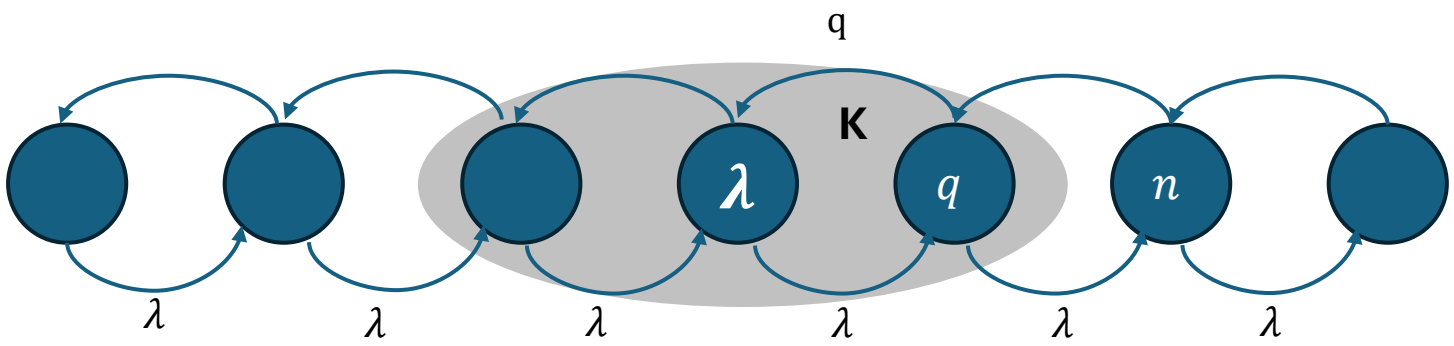
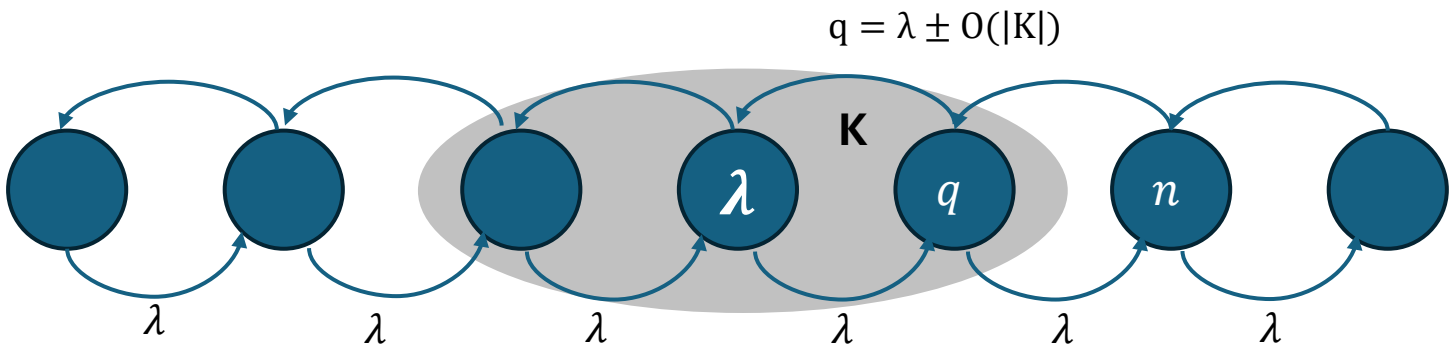


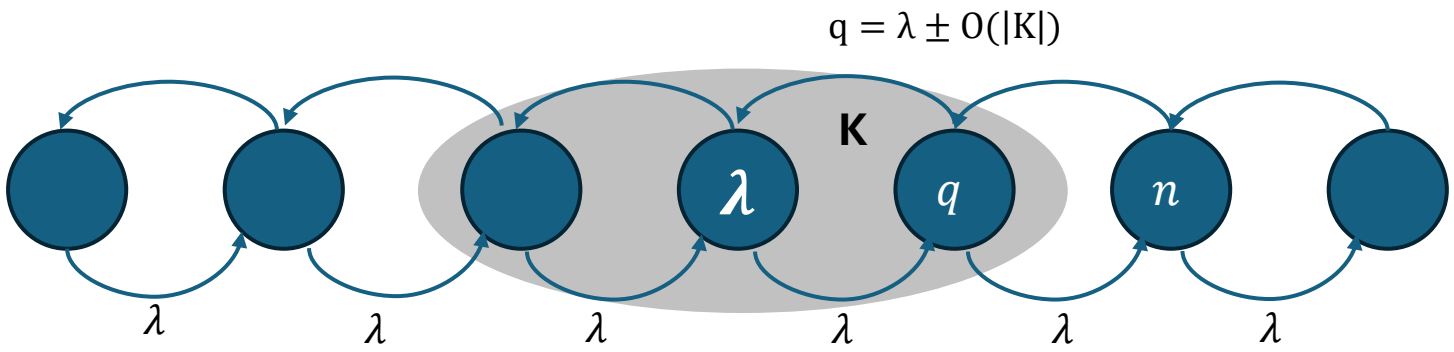
$q \in K$

Step 2: Local mixing



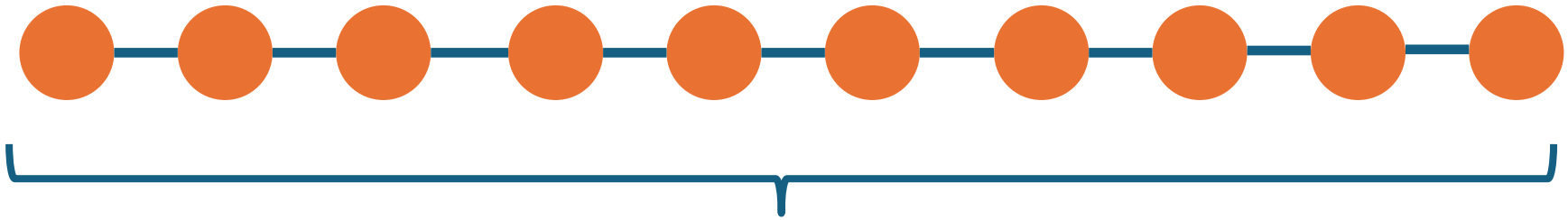
$$|K| = O(n^{1-\alpha})$$

Step 2: Local mixing



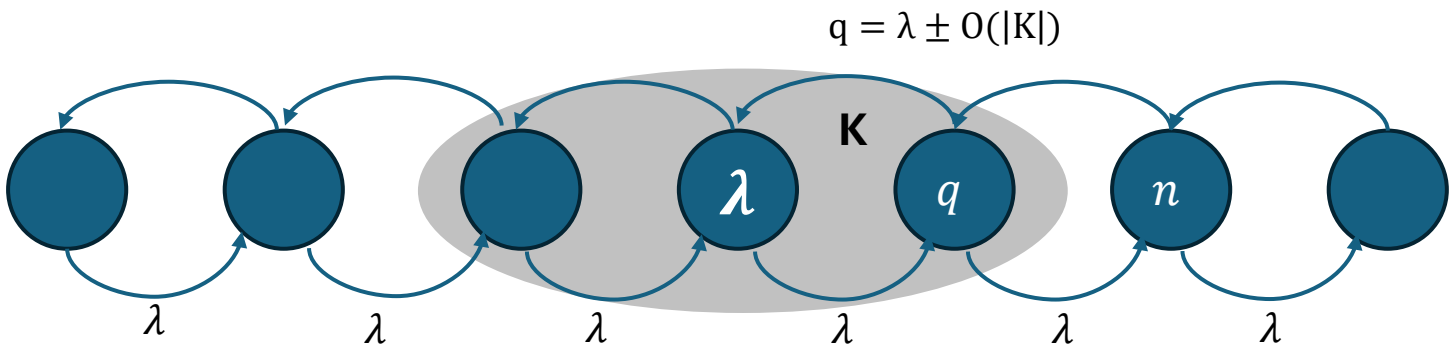
$$|K| = O(n^{1-\alpha})$$

Observation: The system is an "almost" symmetric random walk with a $\Theta(n)$ transition rate inside K



Diameter $|K| = O(n^{1-\alpha})$

Step 2: Local mixing



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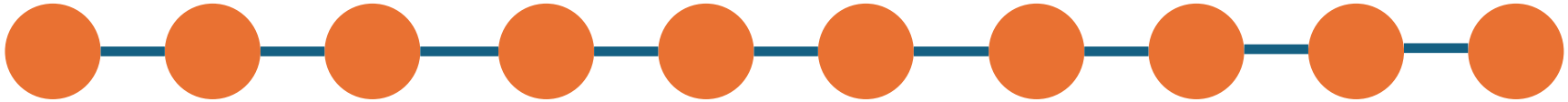
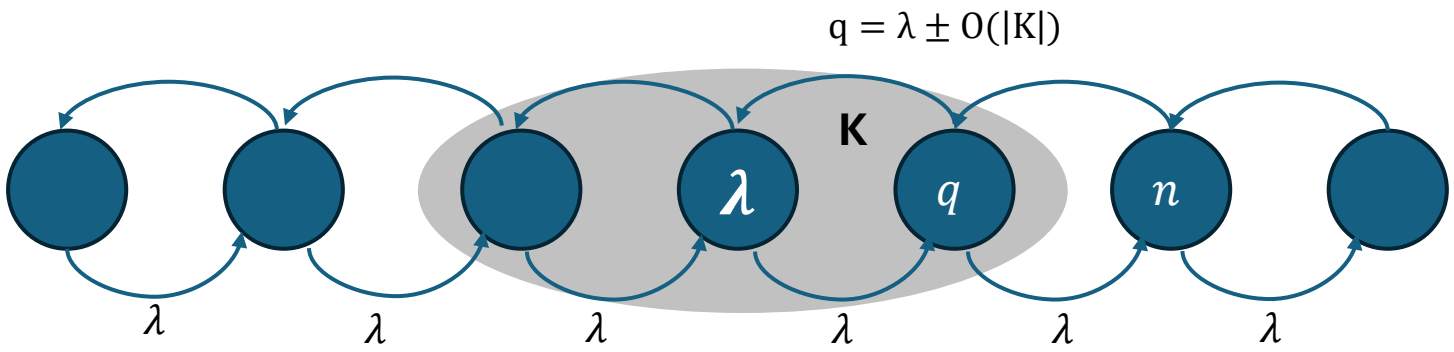


Canonical path method

Step 2: Local mixing



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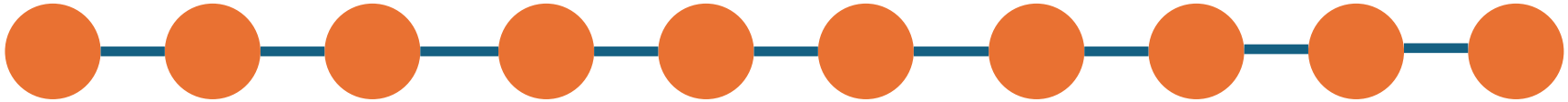
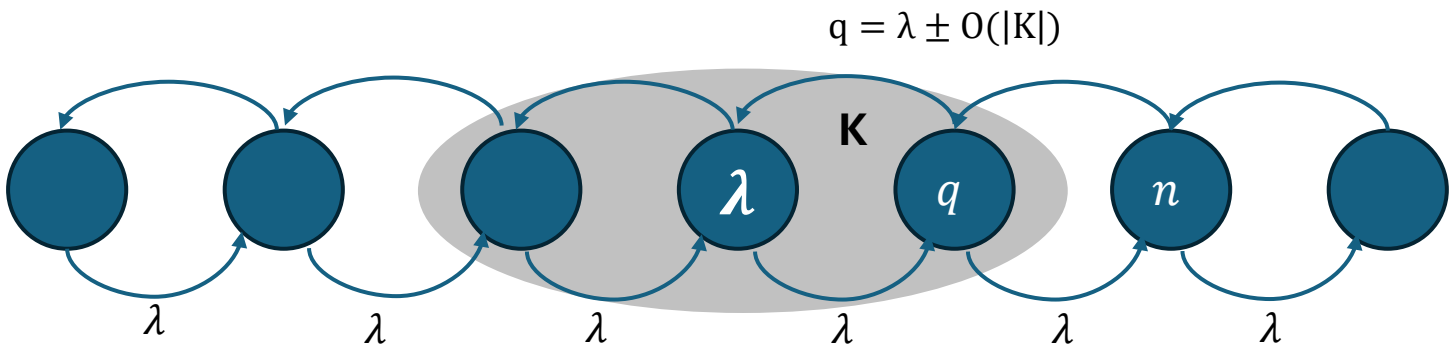


Canonical path method → Congestion factor $O(|K|^2)$

Step 2: Local mixing



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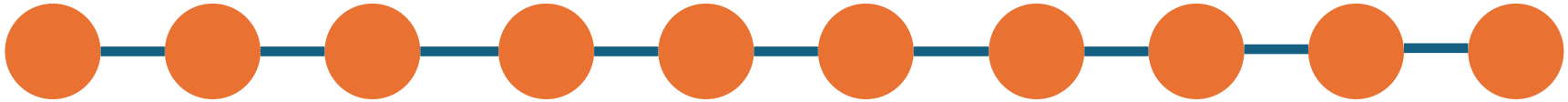
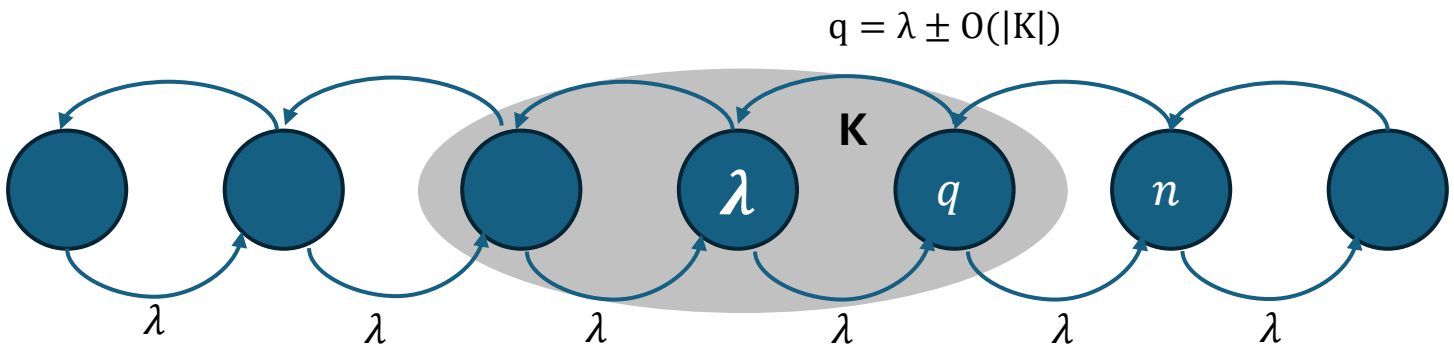


Canonical path method → Congestion factor $O(|K|^2)$ + $\approx n$ transitions per time unit

Step 2: Local mixing



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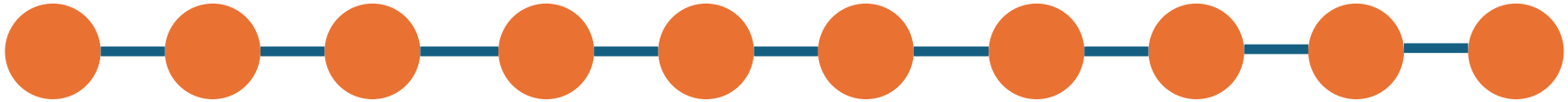
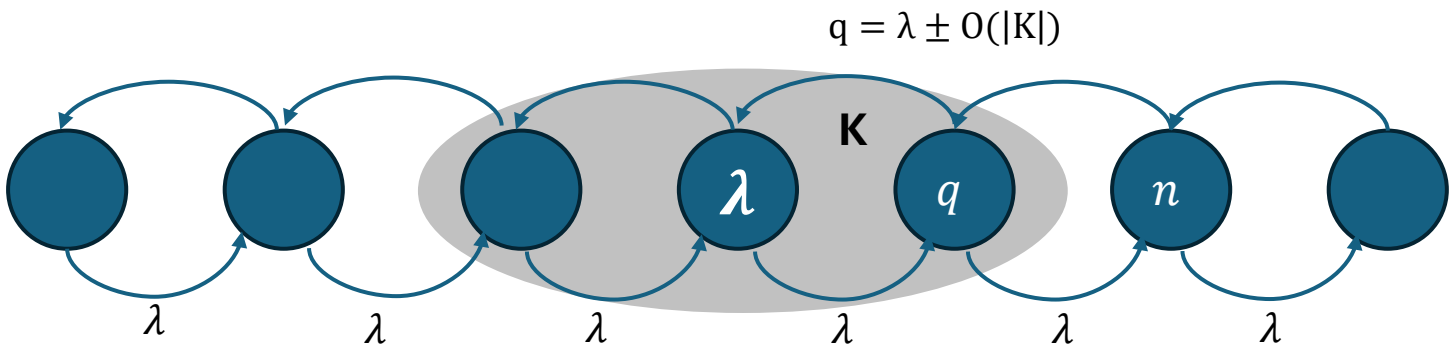


Canonical path method → Congestion factor $O(|K|^2)$ + $\approx n$ transitions per time unit

↓

CTMC mixing time $O\left(\frac{|K|^2}{n}\right)$

Step 2: Local mixing



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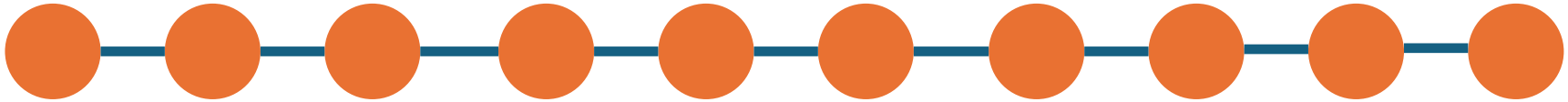

Canonical path method → Congestion factor $O(|K|^2)$ + $\approx n$ transitions per time unit

↓

$\alpha > 0.5$

CTMC mixing time $O\left(\frac{|K|^2}{n}\right) = O(n^{1-2\alpha}) \to 0$

Proof framework: Super-HW regime

Step 1: Lyapunov drift

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$$\mathfrak{Rate} \ge (1-o(1))\left(\sqrt{n}-\sqrt{\lambda}\right)^2$$

Mixing results: All regimes

$$d_\chi(p(t),\pi) \le e^{-\mathfrak{Rate} \times t} d_\chi(p(0),\pi)$$

System	$M/M/\infty$	$\alpha \in (0,1/2)$	$\alpha \in (1/2,\infty)$	$M/M/1$
$\mathfrak{Rate}$	1		$C(\sqrt{n}-\sqrt{\lambda})^2$	$(\sqrt{\mu}-\sqrt{\lambda})^2$
Lyapunov function			**Two-sided Exponential**	**Exp**

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Local set technique			**Canonical path method**	-

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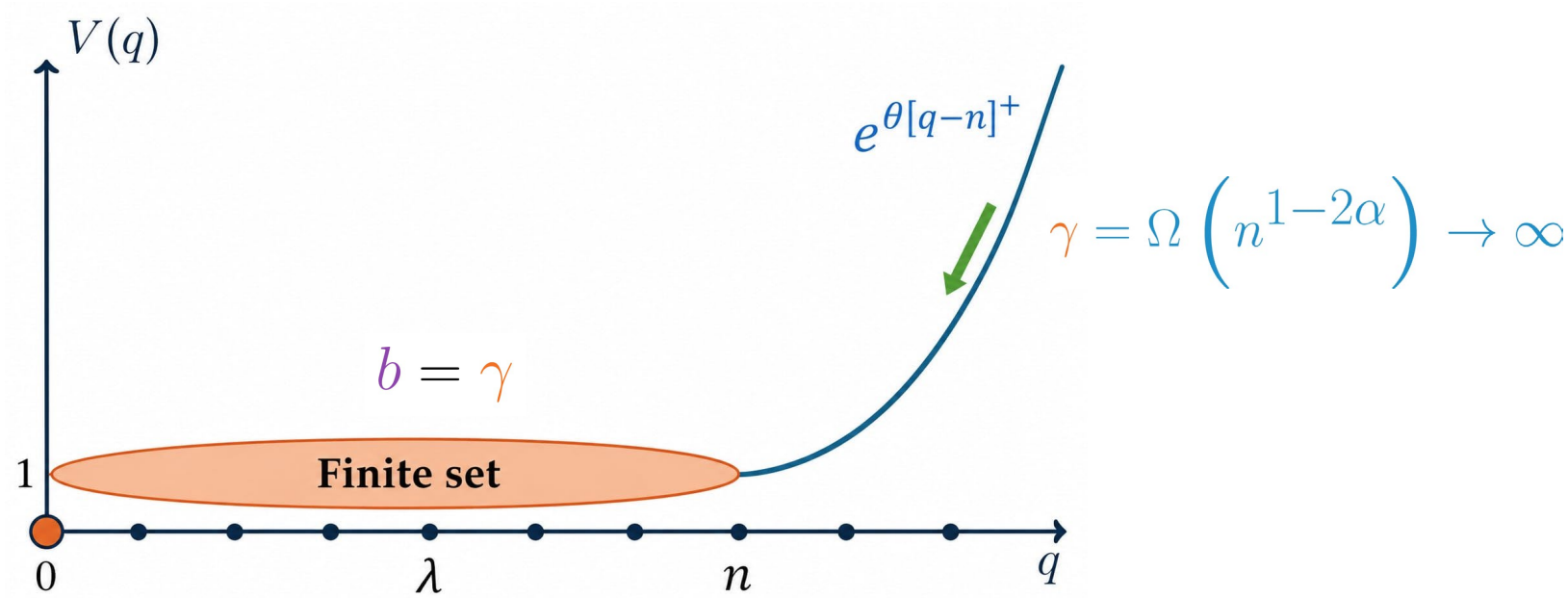

$K = \{0, 1, \ldots, n\}, \qquad V(q) = \exp\{\theta[q-n]^+\}$

Proof framework: Sub-HW regime

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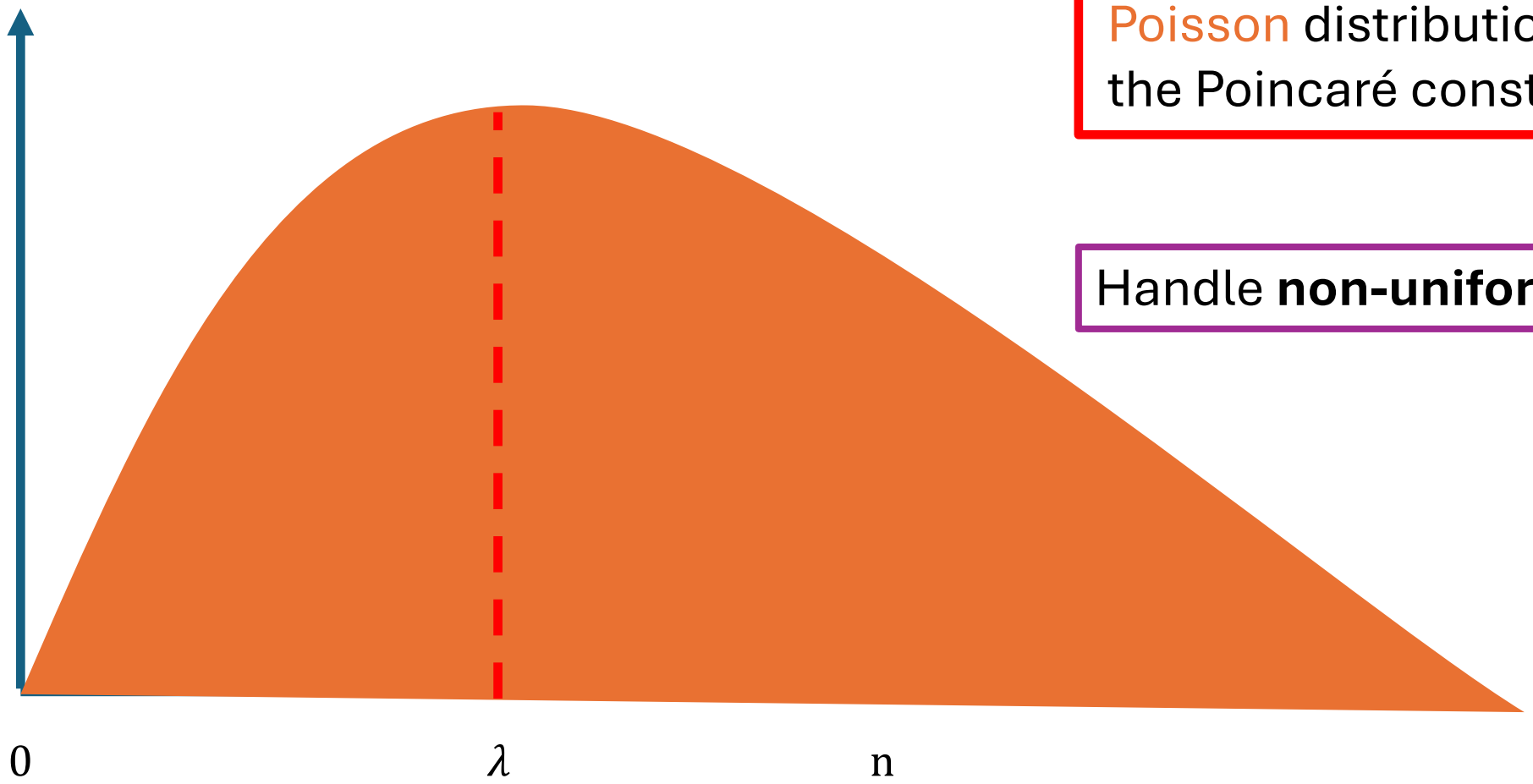
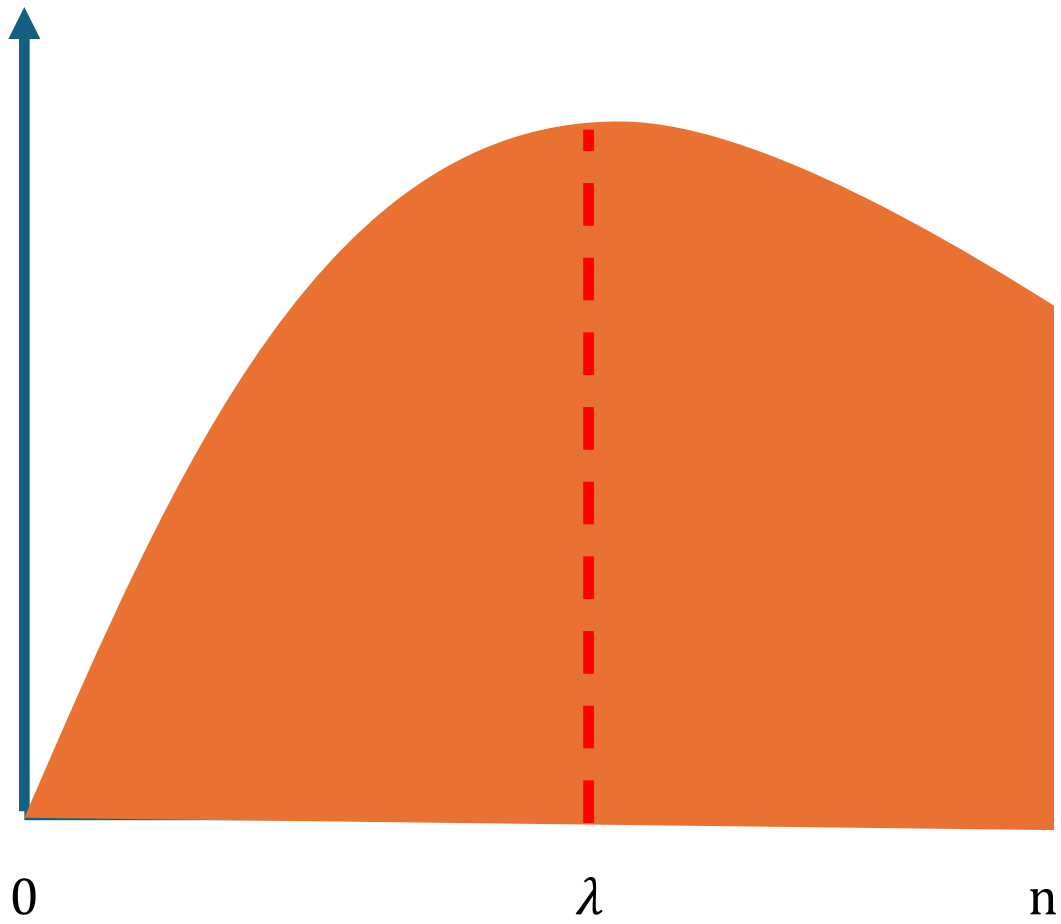
Step 2: Local mixing bound



Poisson distribution admits the Poincaré constant 1

Handle **non-uniform** local set!

Step 2: Local mixing bound



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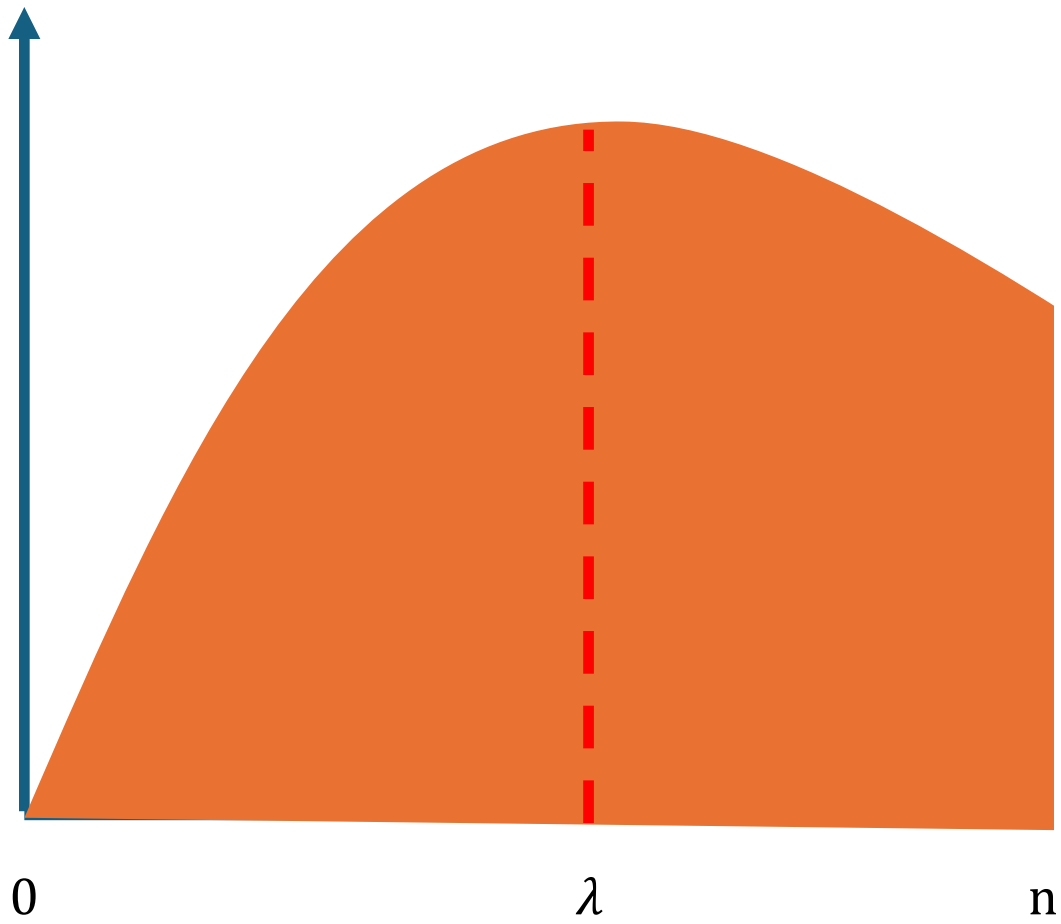

Truncated distribution admits the Poincaré constant 1

Handle **non-uniform** local set!

Proof framework: Sub-HW regime

Step 1: Lyapunov drift

$$\mathcal{L}V \leq -\Omega(n^{1-2\alpha})V + b\,1_K$$

Step 2: Local mixing

$$\mathrm{Var}_{\pi_K}(f) \leq \frac{C_L}{\pi(K)}\langle f, -\mathcal{L}f\rangle_\pi$$

Step 3: $C_P = \dfrac{1 + bC_L}{\gamma}$

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$$\mathfrak{Rate} \ge 1 - o(1)$$

Mixing results: All regimes

$$\chi(q(t), q(\infty)) \leq e^{-\mathfrak{Rate} \times t} \chi(q(0), q(\infty))$$

System	$M/M/\infty$	$\alpha \in (0, 1/2)$	$\alpha \in (1/2, \infty)$	$M/M/1$
$\mathfrak{Rate}$	1	$\Omega(1)$	$C(\sqrt{n} - \sqrt{\lambda})^2$	$(\sqrt{\mu} - \sqrt{\lambda})^2$
Lyapunov function	**Linear**	**Constant + Exp**	**Two-sided Exponential**	**Exp**
Local set technique	-	**Truncation method**	**Canonical path method**	-

Tightness of the mixing rate

Non-asymptotic rate

Theorem 1 [NVM 2025]

Sub-Halfin-Whitt regime

$$\mathfrak{Rate} = \Omega(1)$$

$n \to \infty$

Super-Halfin-Whitt regime

$$\mathfrak{Rate} = C(\sqrt{n} - \sqrt{\lambda})^2$$

$n \to \infty$

Limiting rate

Theorem 2 [NVM 2025]

$$\mathfrak{Rate} \to 1$$

$$\frac{\mathfrak{Rate}}{(\sqrt{n} - \sqrt{\lambda})^2} \to 1$$

Upper bound of mixing rates

Variational upper bound

$$\mathfrak{Rate}_n = \inf_{\mathrm{Var}_{\pi_n}(f)>0} \frac{\langle f, -\mathcal{L}f \rangle_{\pi_n}}{\mathrm{Var}_{\pi_n}(f)}$$

Sub-Halfin-Whitt

Upper bound of mixing rates

Variational upper bound

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Sub-Halfin-Whitt

Test function $f(q) = q$

Upper bound of mixing rates

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Sub-Halfin-Whitt

Test function $f(q) = q$

$$\langle f, -\mathcal{L}f \rangle_{\pi_n} = \sum_{q=0}^{\infty} \lambda \pi_n(q) \left(f(q+1) - f(q)\right)^2 = \lambda$$

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Sub-Halfin-Whitt

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Test function $f(q) = q$

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Super-Halfin-Whitt

Test function $f_a(q) = \begin{cases} 1, & q < n, \\ a^{q-n}, & q \ge n, \end{cases} \qquad 1 < a < \sqrt{n/\lambda}$

Upper bound of mixing rates

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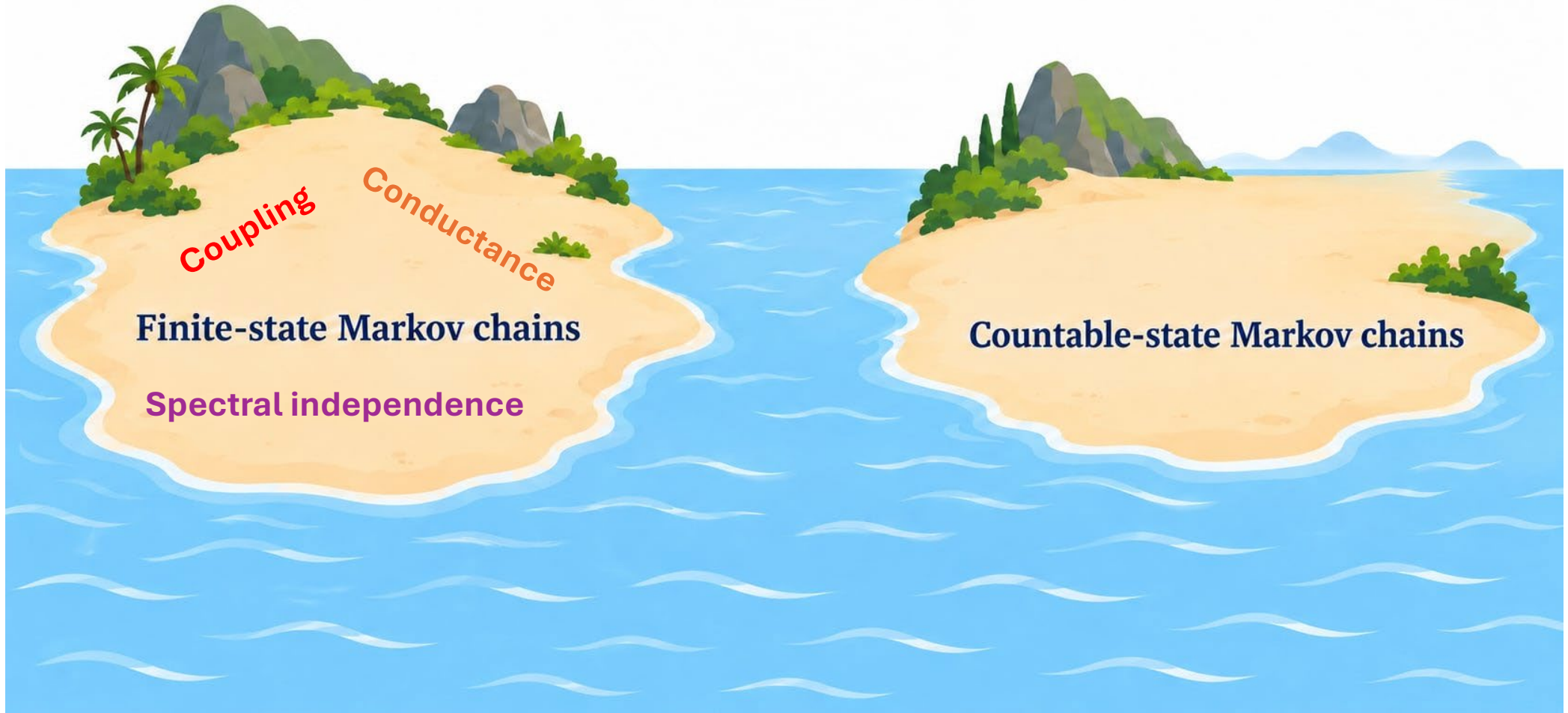
Takeaway


Coupling
Finite-state Markov chains
Countable-state Markov chains

Takeaway


Coupling
Conductance
Finite-state Markov chains
Countable-state Markov chains

Takeaway


Coupling
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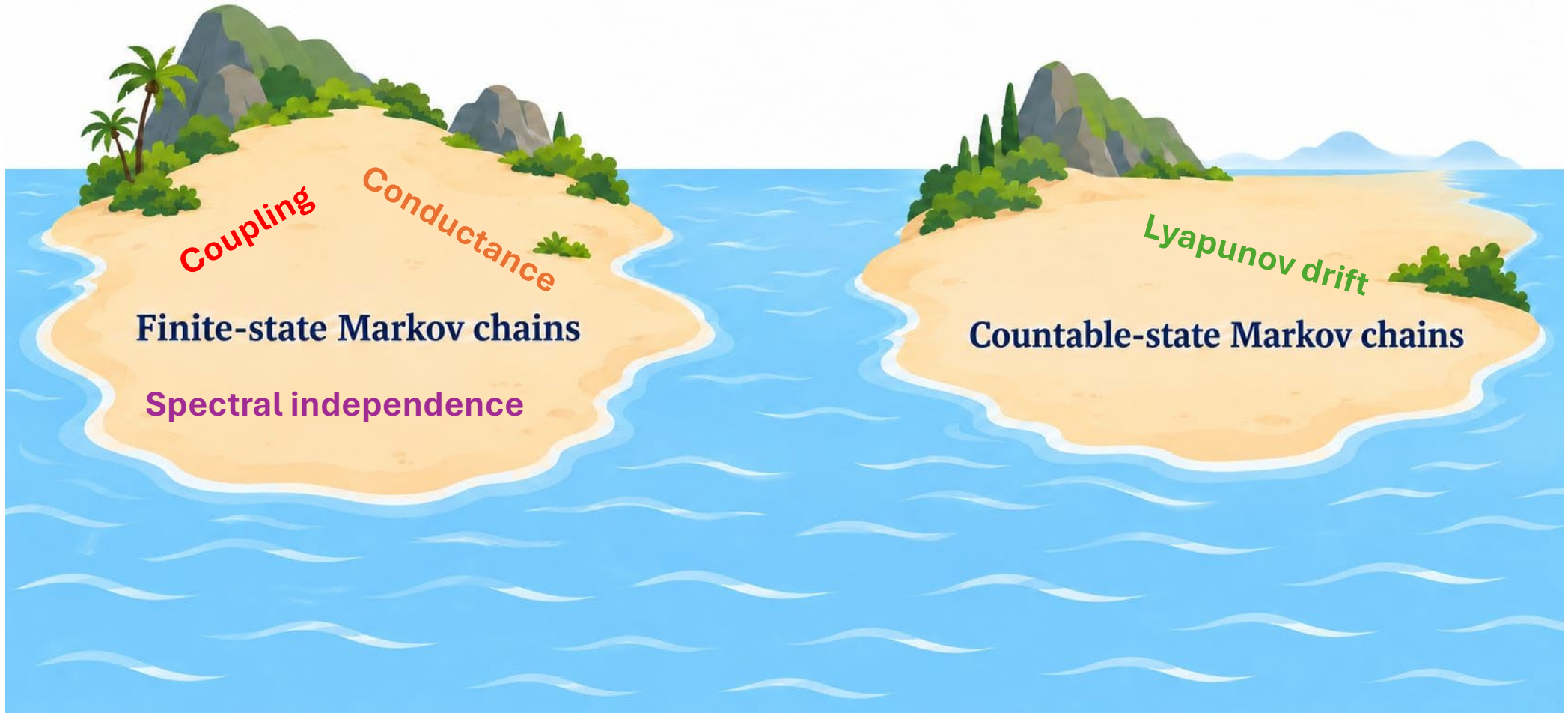
Coupling
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Takeaway


Lyapunov-Poincaré
Coupling
Conductance
Finite-state Markov chains
Spectral independence
Lyapunov drift
Countable-state Markov chains

Takeaway

Applicable to **reversible** Markov chains!



Takeaway

Applicable to **reversible** Markov chains!

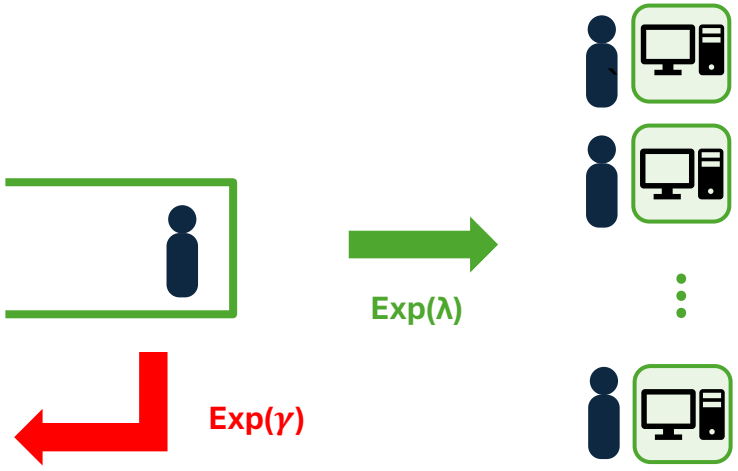
Unifying Lyapunov-Poincaré method!



Future work: Reversible Markov chains

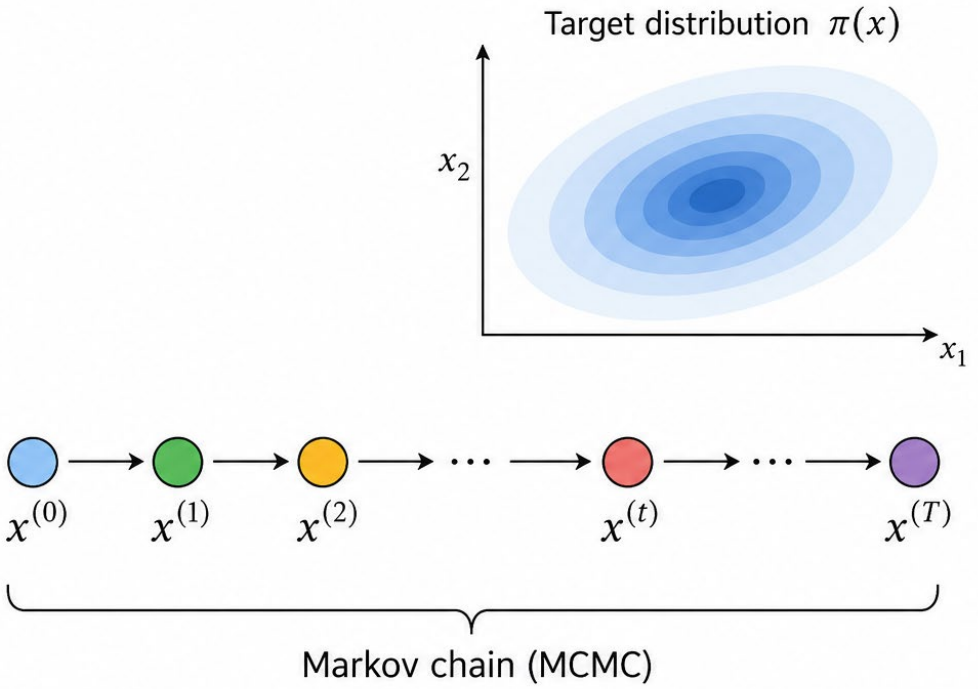
Abandonment queue

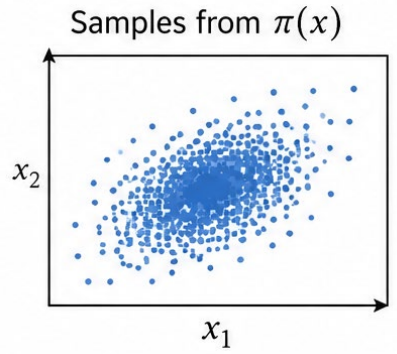
Forthcoming



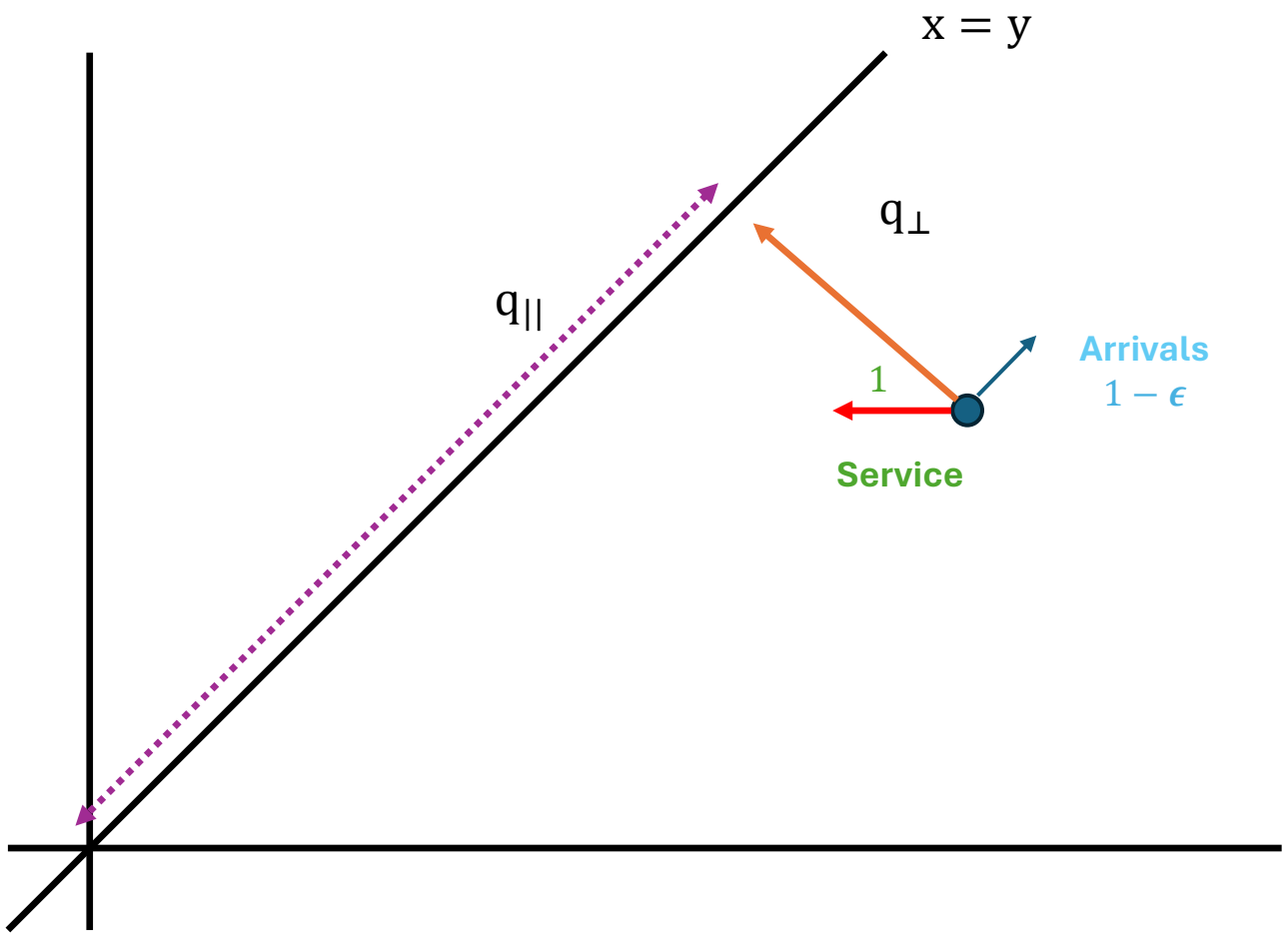
Curvature method

Sampling on infinite state space



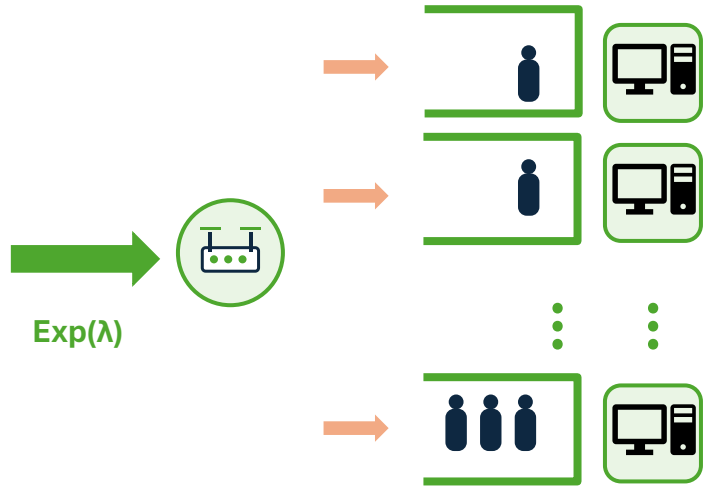


Future work: Non-reversible Markov chains



Load-balancing system

Ongoing



Coupling method

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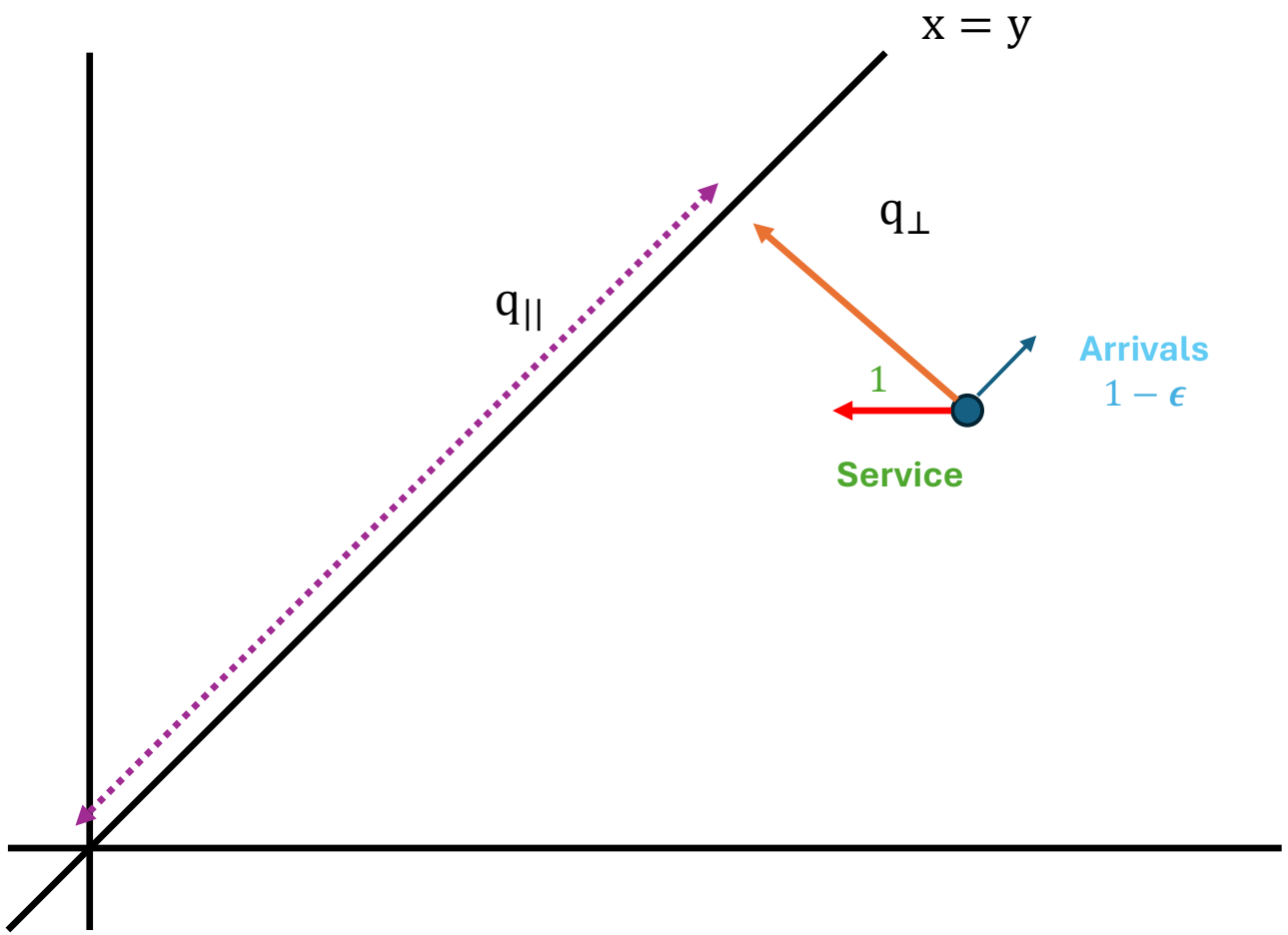

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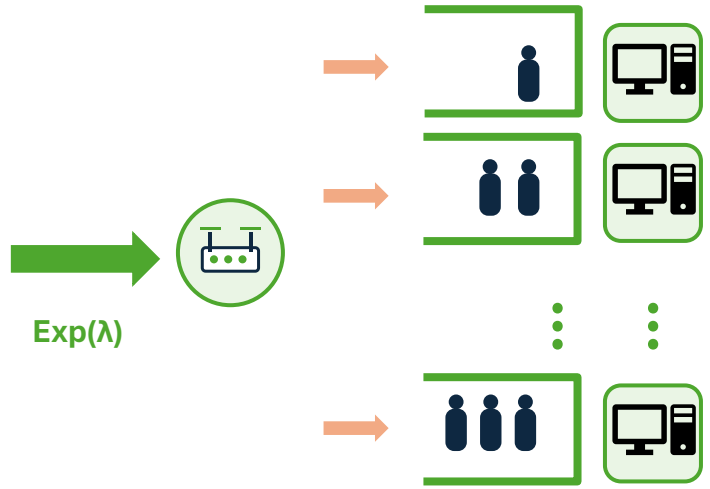

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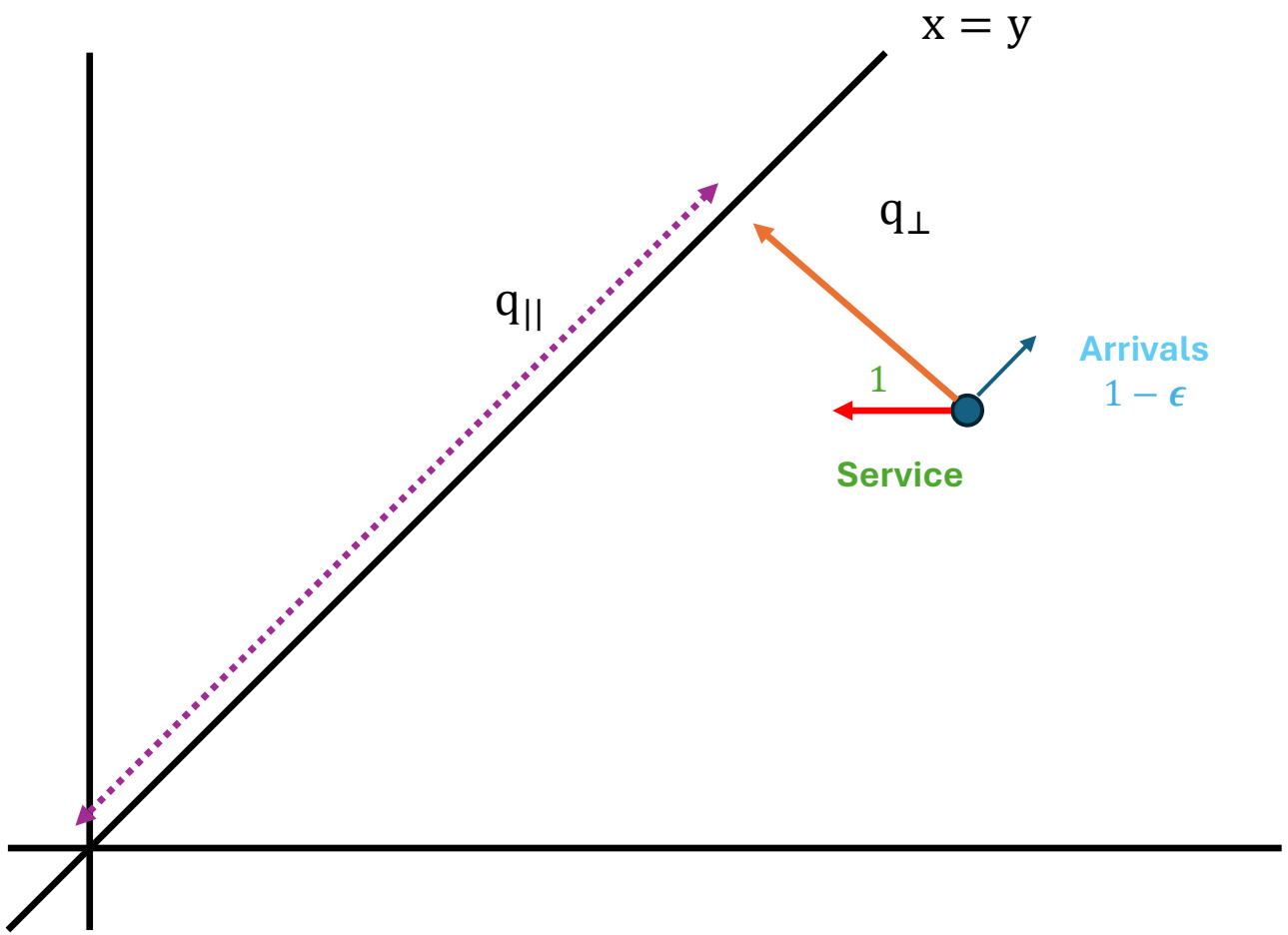

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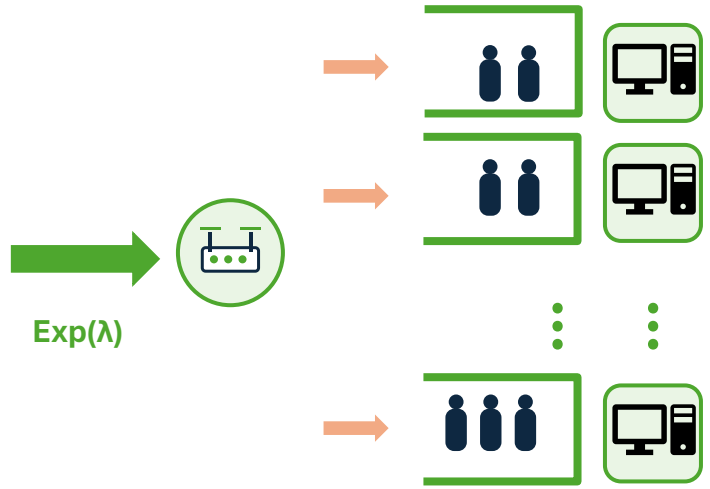
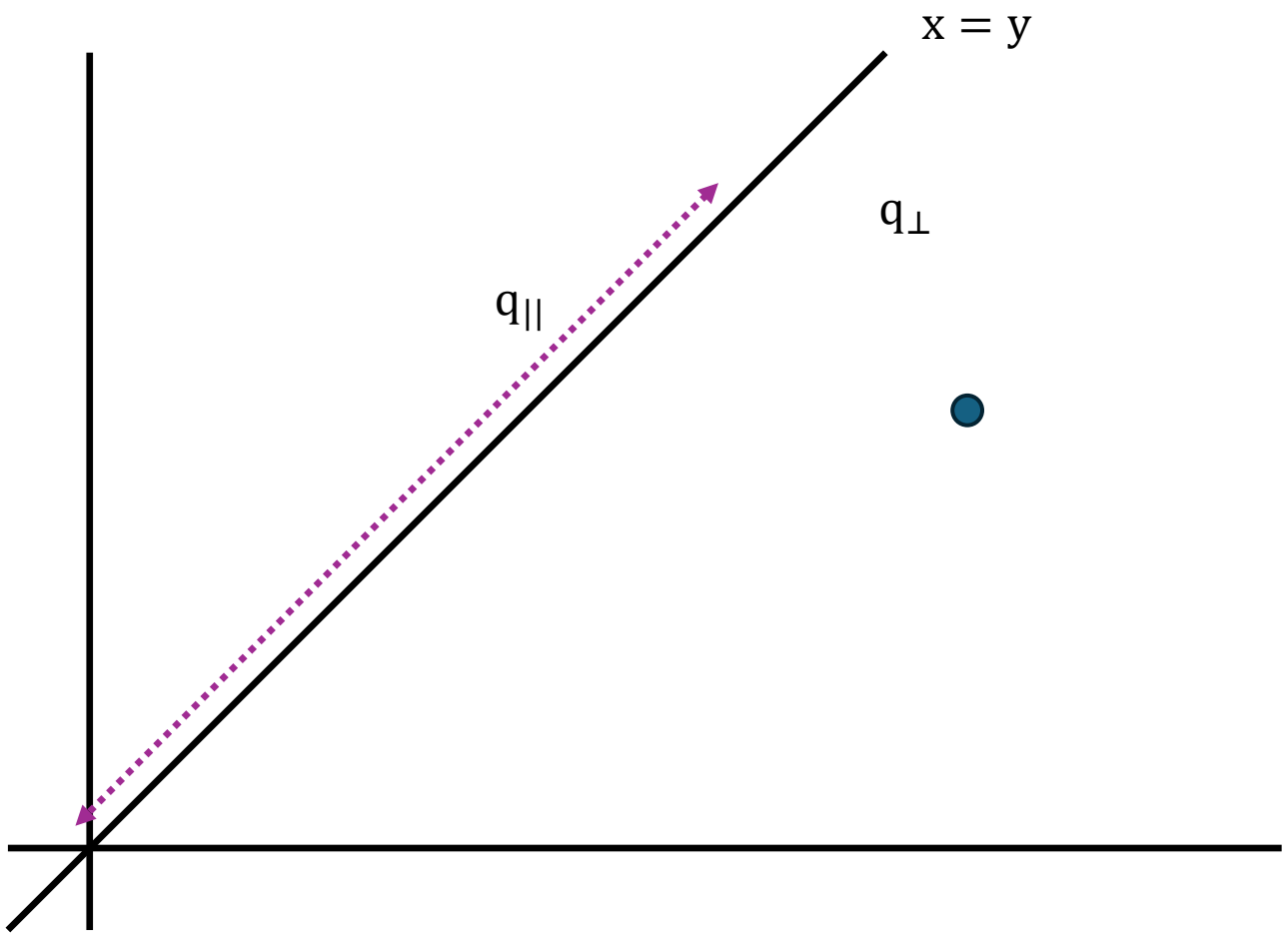
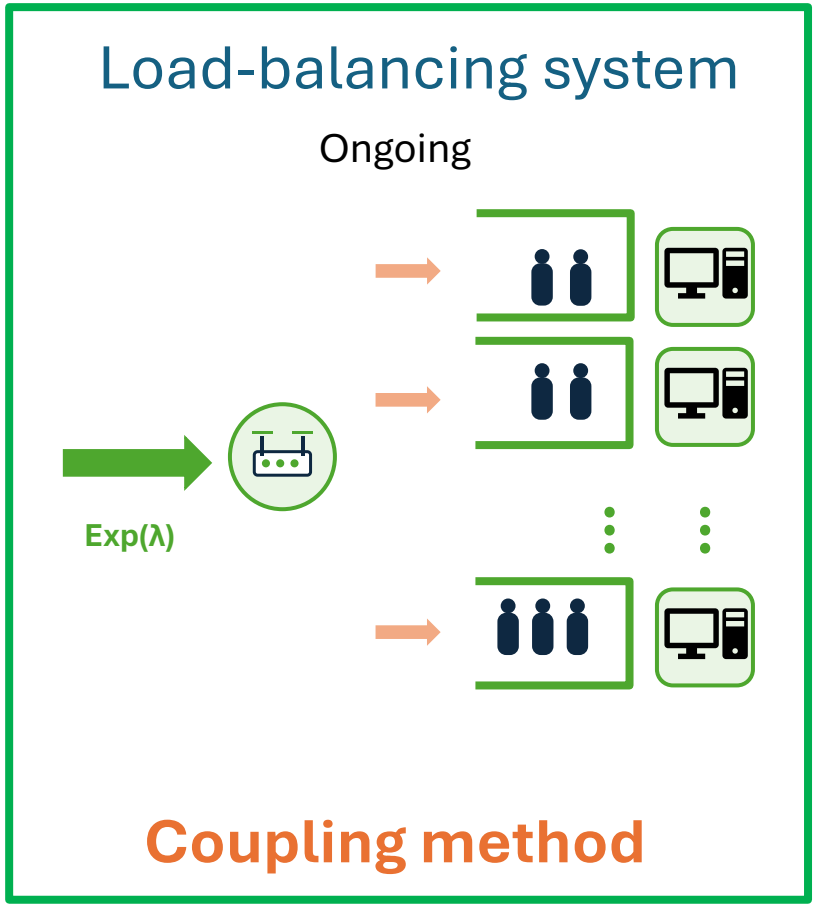


Coupling method

Future work: Non-reversible Markov chains





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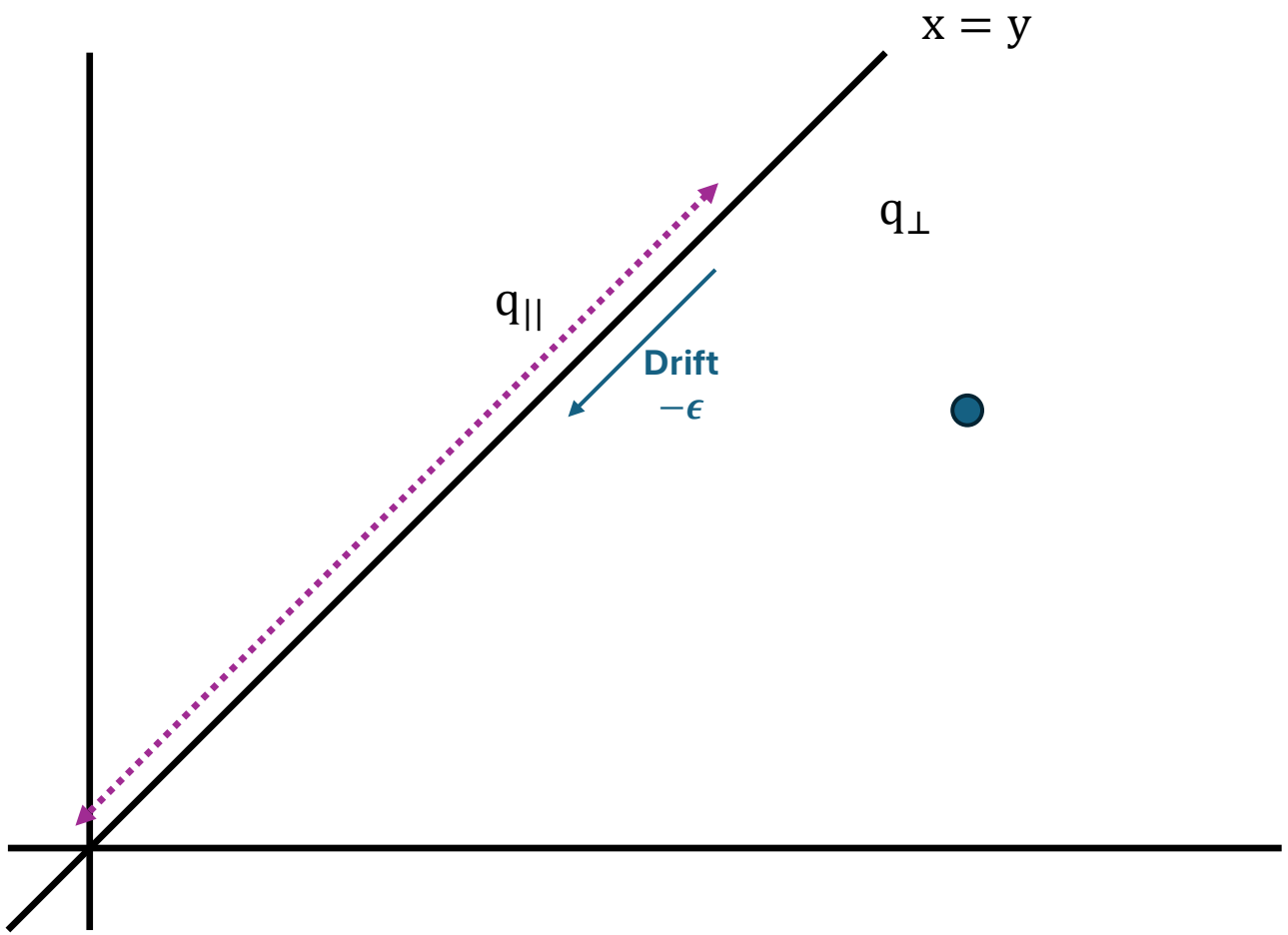

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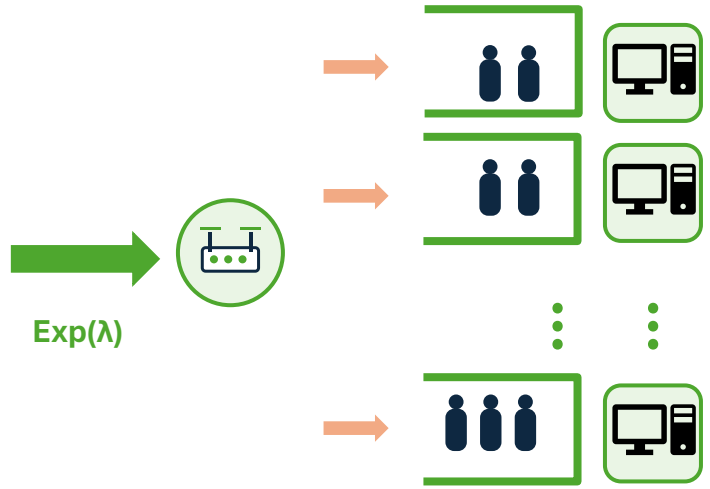
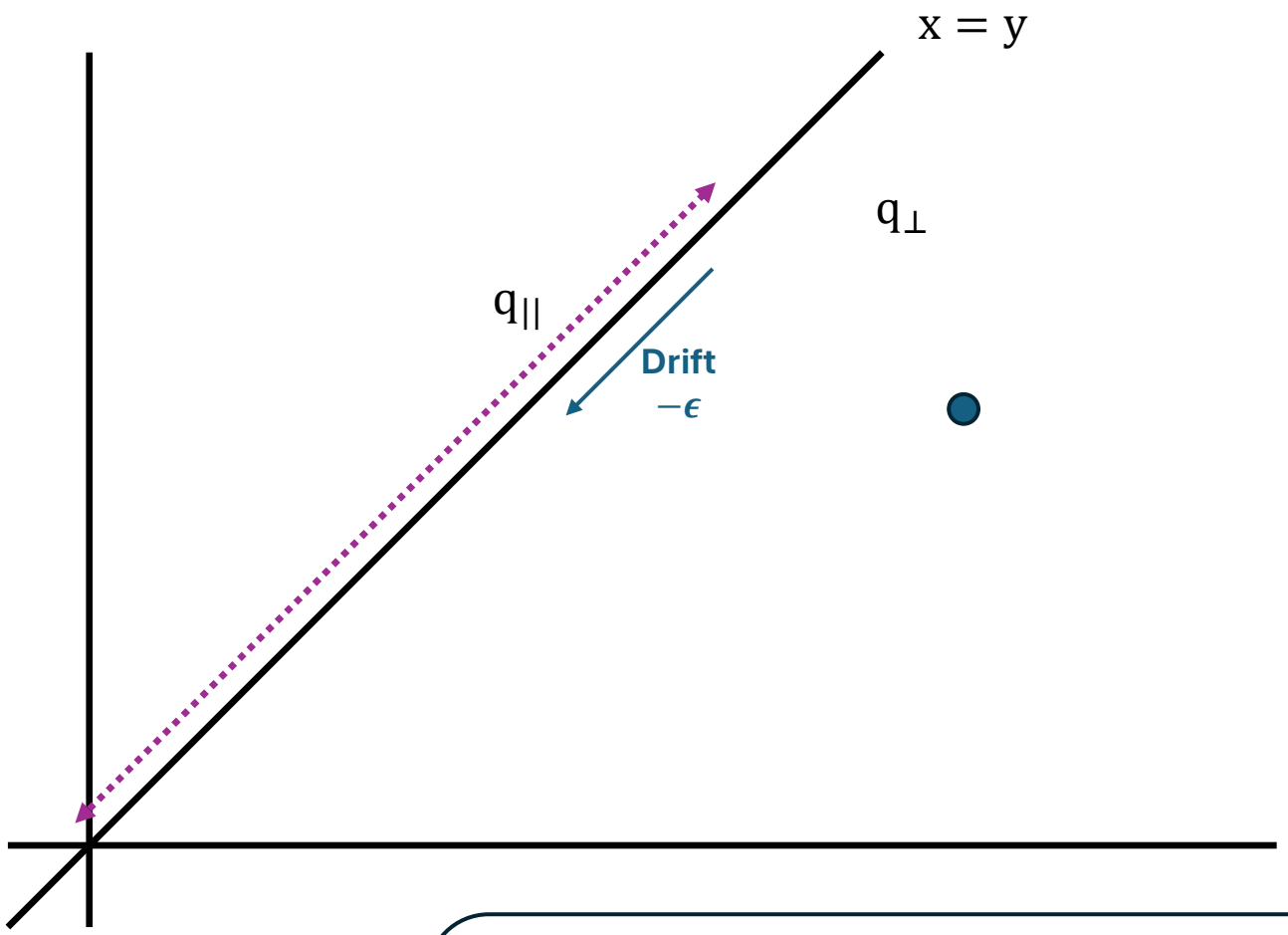
Ongoing



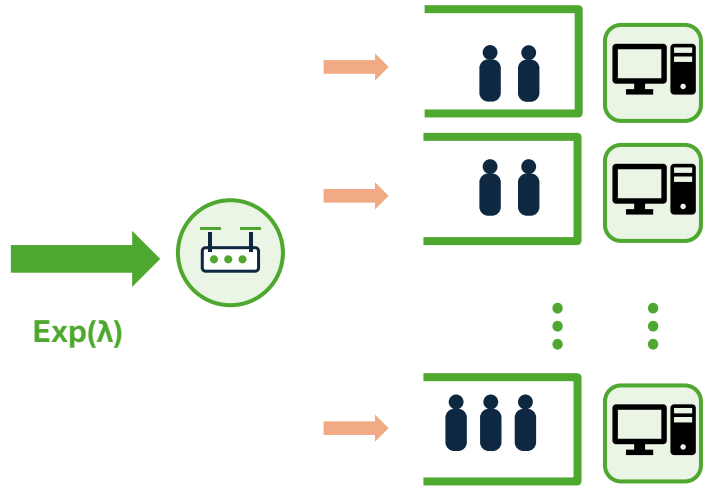
Coupling method

Future work: Non-reversible Markov chains



Load-balancing system

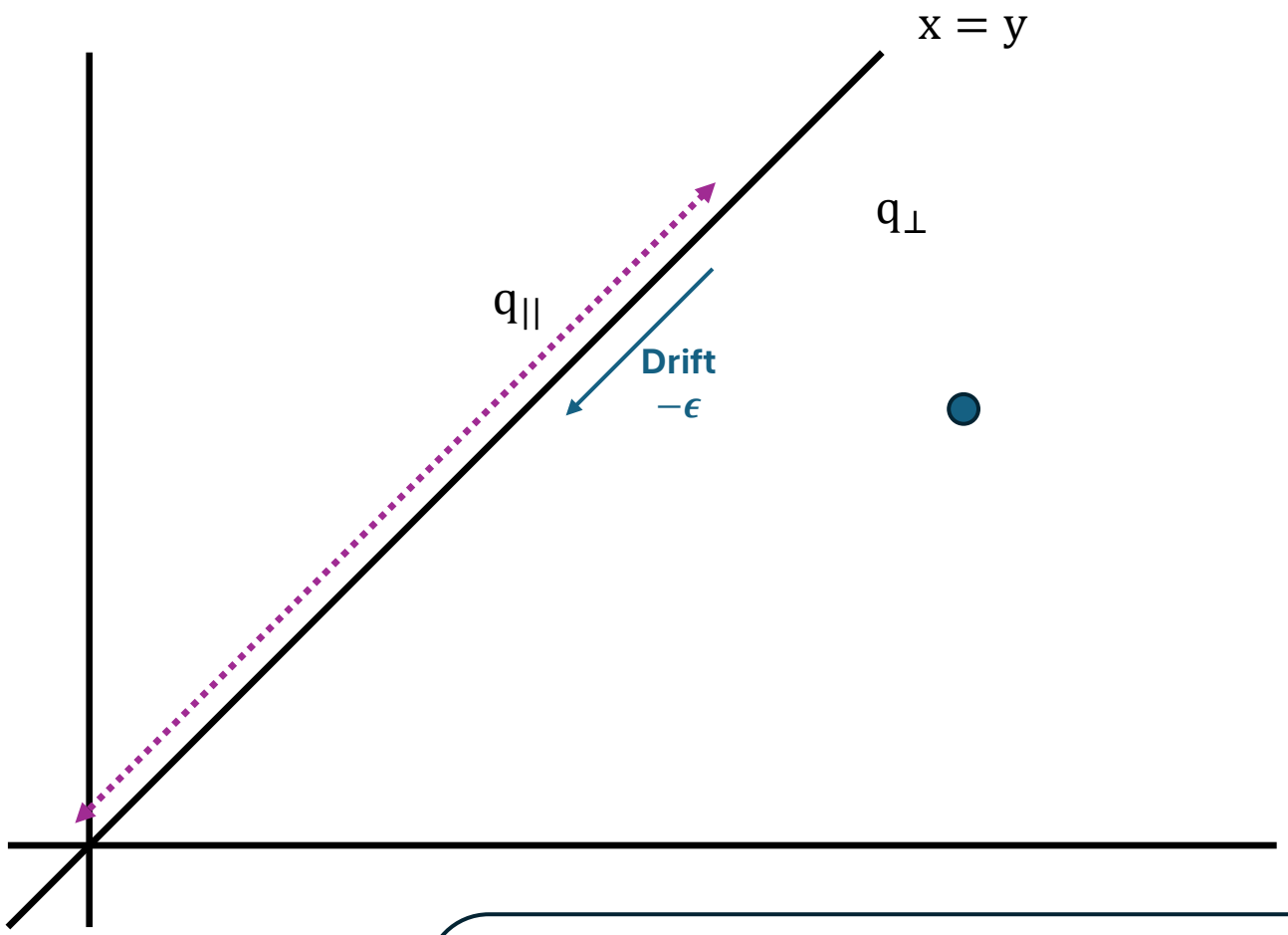
Ongoing



Coupling method

Challenges to overcome

Future work: Non-reversible Markov chains



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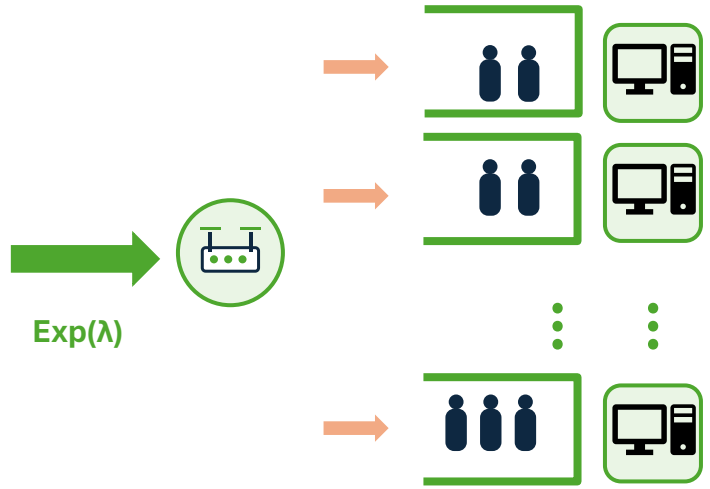

Coupling method

Challenges to overcome

Two timescale drift

Non-reversibility

Thank you!