Influence Enhancement in Opinion Dynamics Using Edge Modification: A Kron Reduction-Based Approach

Twinkle Tripathy

Joint work with Aashi Shrinate, Aravind Seshadri, Laxmidhar Behera,

Lingfei Wang and Karl H. Johansson

Indian Institute of Technology, Kanpur

October 27, 2026

Overview

1. Introduction and Motivation
2. Problem Formulation: Influence Enhancement Through Edge Modifications
3. Impact of an Edge Modification
4. Topology-Based Edge Modifications
5. Kron Reduction Framework for Influence Enhancement
6. Conclusions

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Opinion dynamics

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Preliminaries: opinion evolution models

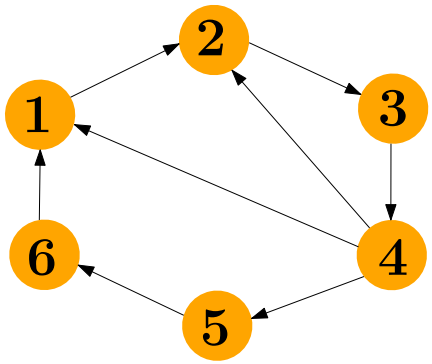


Figure: Graphical representation of interacting individuals

DeGroot's Model[5]

- Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V}$: agents and $\mathcal{E}$: interactions

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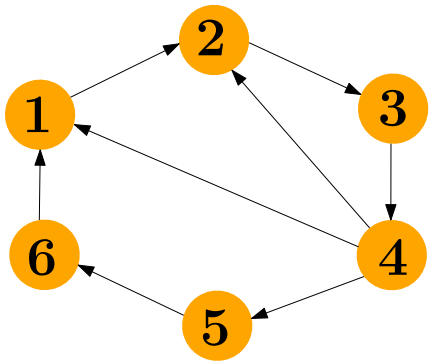


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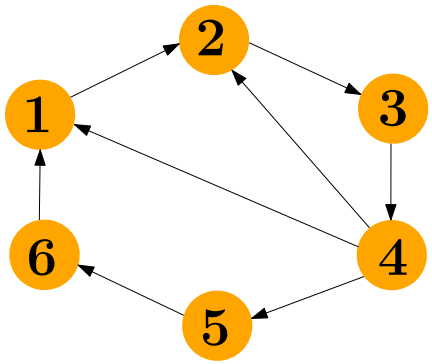


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- $W = [w_{ij}]$: adjacency matrix of $\mathcal{G}$
  - $w_{ij} > 0$ only if $(j, i) \in \mathcal{E}$
  - W is row-stochastic

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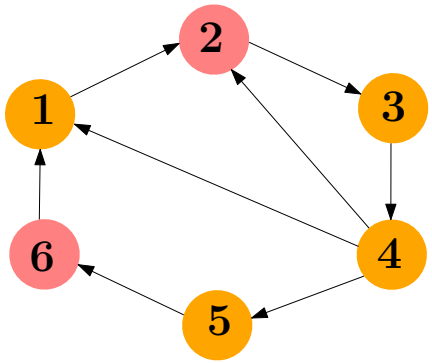


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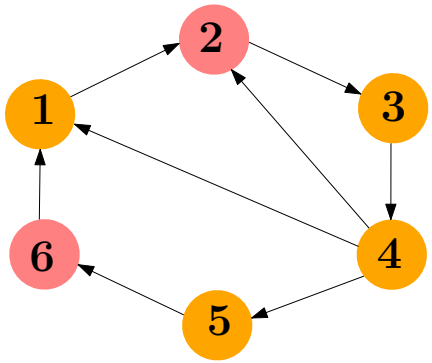


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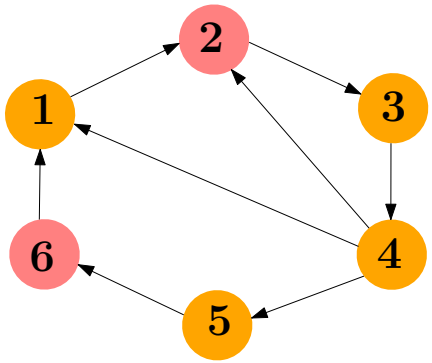


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- Stubbornness matrix $\beta = diag(\beta_1, ..., \beta_n)$; $\quad \beta_i \in [0, 1]$ for $i \in \{1, 2, ..., n\}$
- Explains persistent disagreement; better fit to human-subject experiments than Bayesian models.

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Steady state behaviours and influence centrality

Assumption 1

$\mathcal{G}$ is weakly connected with $m > 1$ stubborn agents, and each non-stubborn agent has a path from at least one stubborn agent.

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- $\mathbf{c}$ depends on:
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  - Network Interactions (W)

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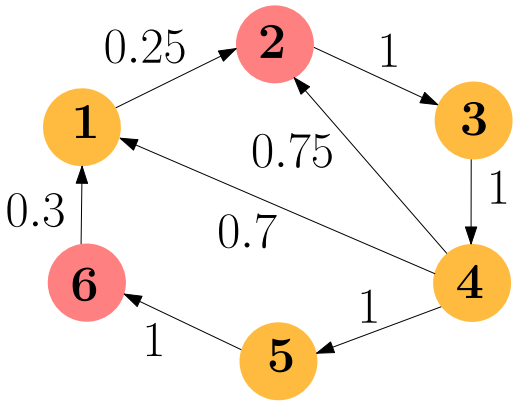
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Influence Enhancement Through Edge Modifications

Problem Formulation

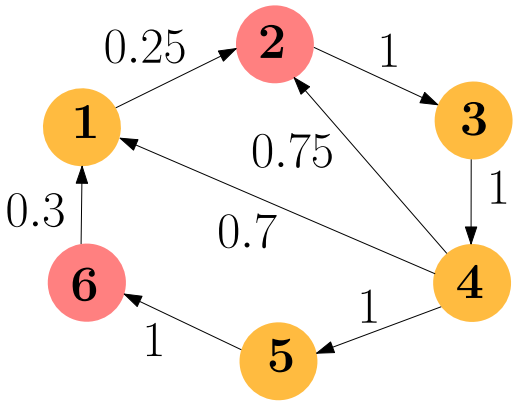
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- There are agents who want to increase their influence.
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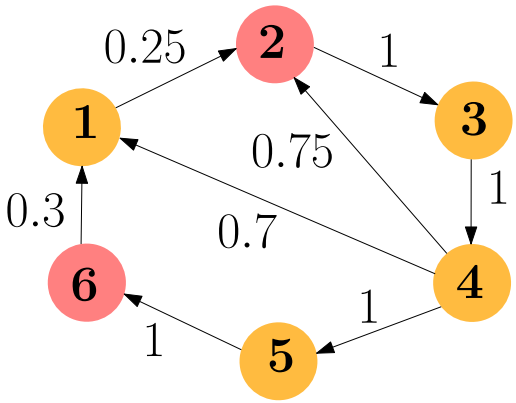
Feasible solutions

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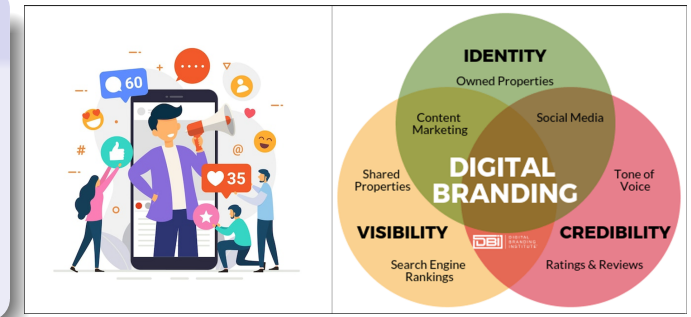


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Real world examples of edge modifications

- design of social media feeds,
- targeted marketing campaigns,
- coalition formations in politics, diplomacy, *etc.*



Solution 1: increasing influence by changing stubbornness

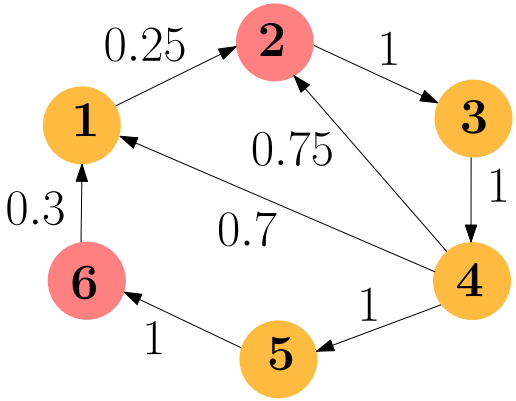


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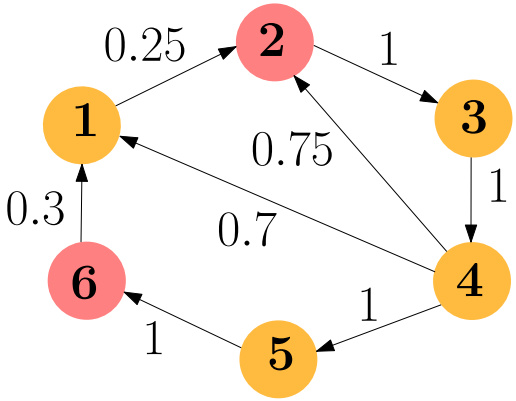


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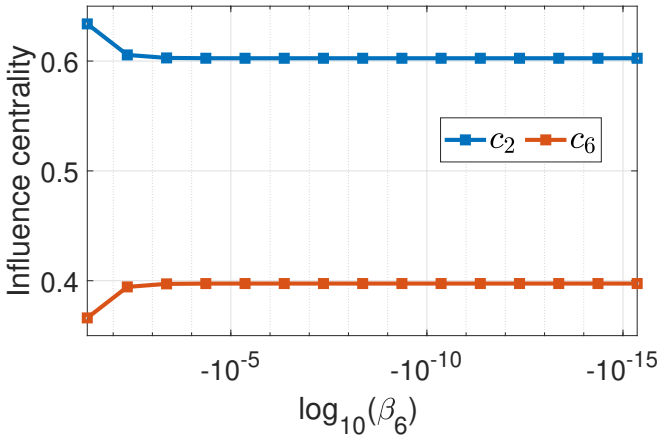


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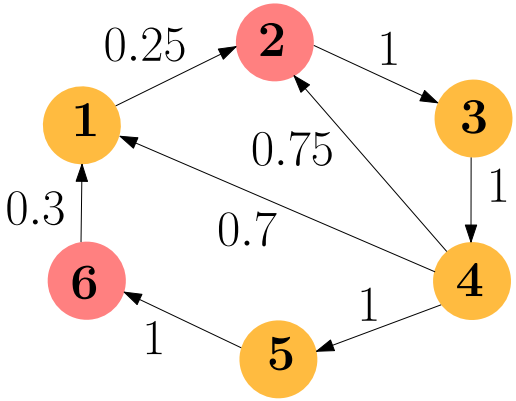


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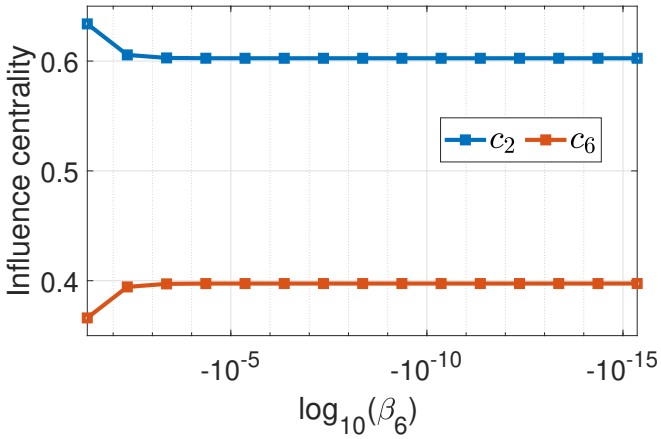


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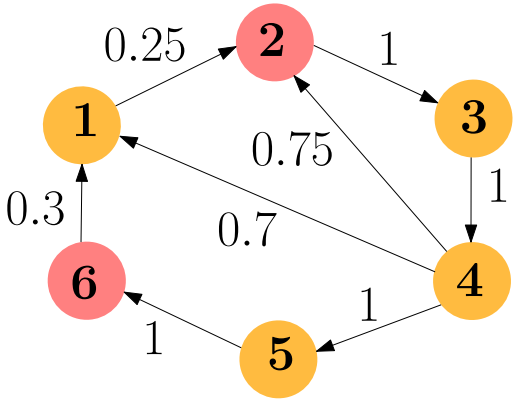


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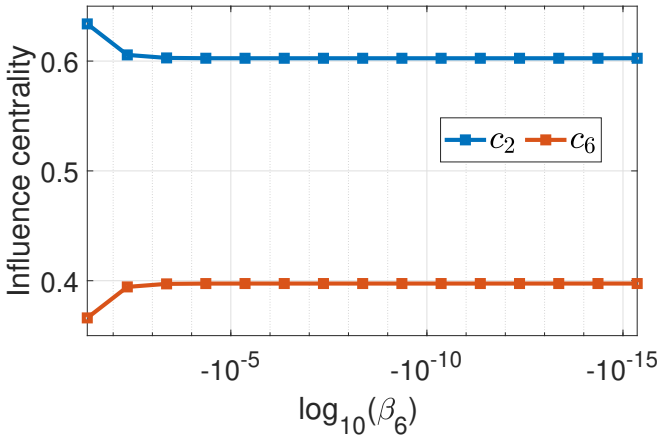


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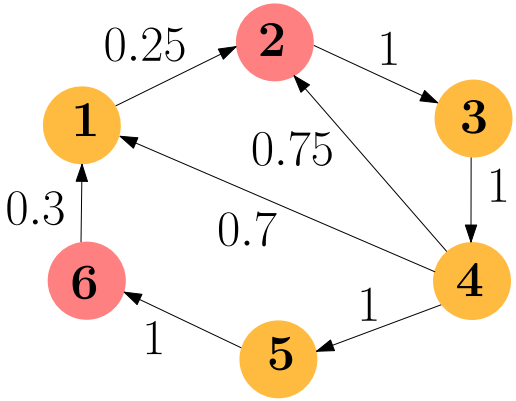


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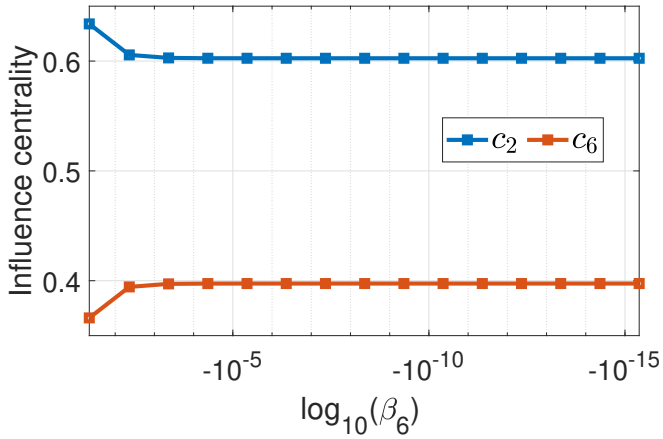


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- **Solution 1** does not always work.
- Stubbornness is an inherent trait: often infeasible to modify, and not always effective.
- **Network interventions** are practical: feed and recommendation algorithms, targeted marketing, coalition formation.

Solution 2: increasing influence through edge modifications

Existing edge-addition approaches

- Ancona et al. (pro-vaccine campaign), Ao and Jia (two-agent game), Wang et al. (influence maximisation).
- Edges are added *from an external stubborn agent* whose opinion is never affected by others, to an undirected-connected or strongly connected network.

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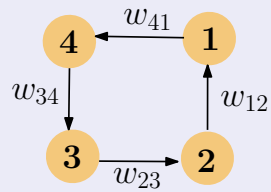
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Limitations and this work

- Adding edges from the desired agent **need not increase** its influence.
- Real world networks are often weakly connected.

Proposed edge modification (a, b, d)

An edge modification (a, b, d) with distinct nodes a, b, d applied to $\mathcal{G}$:



(a) Original network $\mathcal{G}$

Figure: Effect of edge modification (1, 3, 4) on $\mathcal{G}$.

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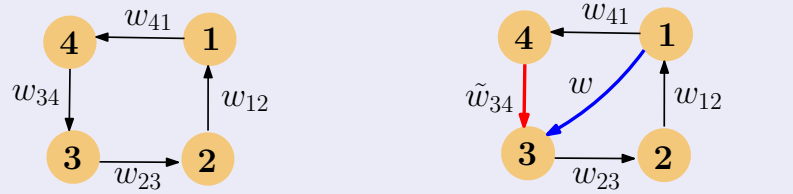
(a) Original network $\mathcal{G}$ **(b)** The addition of (1, 3)

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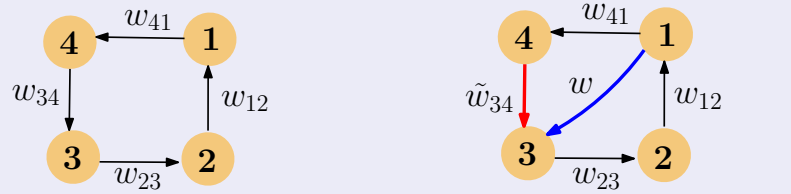
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- Choose $w < w_{b,d}$ (edge (d, b) is not removed, weights stay positive).

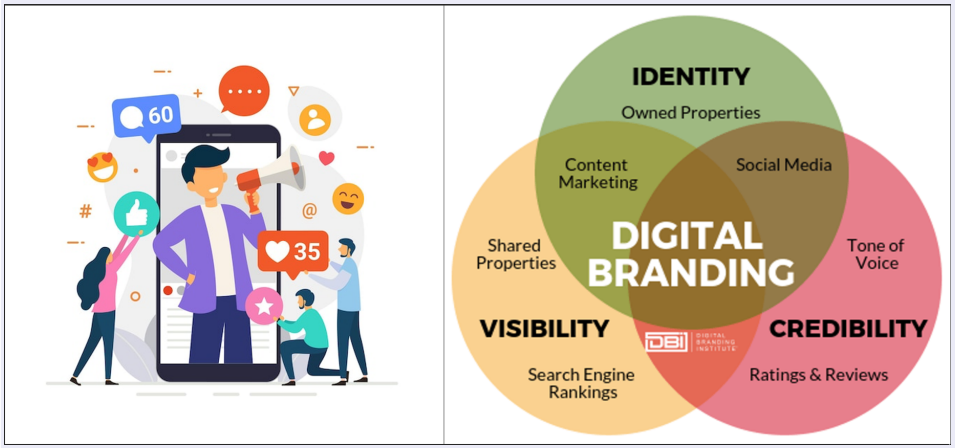


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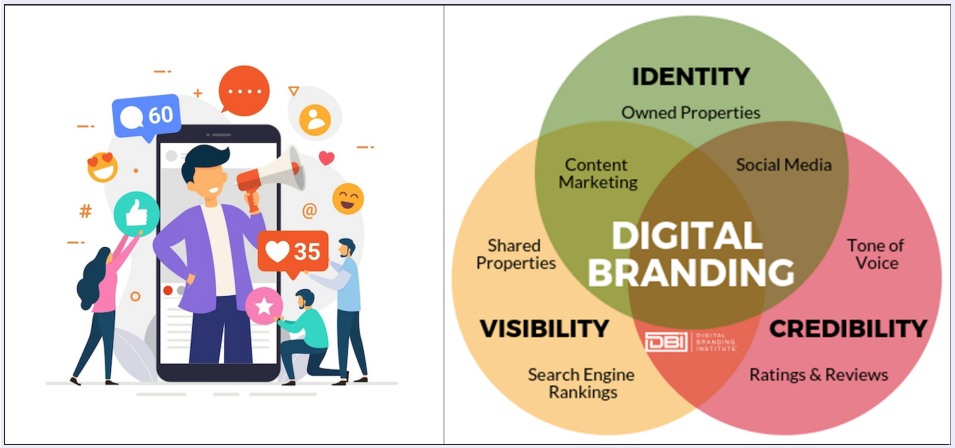
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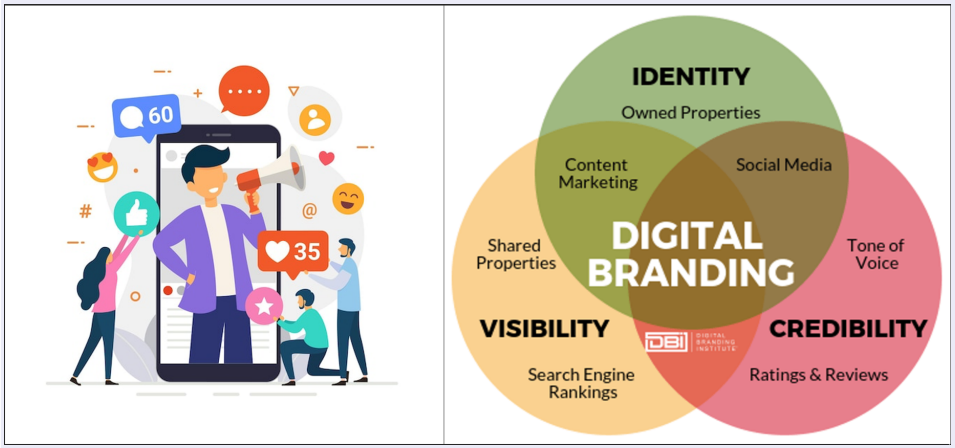
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- Mathematical representation in a graph $\mathcal{G}$ with adjacency matrix W:

$$\hat{W} = W + w\, e_b (e_a - e_d)^T \qquad \text{(rank-one update of the interaction matrix)}$$

Assumptions and problems of interest

Standing assumptions

- **Assumption 1:** each non-stubborn agent has a path from at least one stubborn agent.
- **Assumption 2:** the agent b in (a, b, d) has $\beta_b < 1$ (a fully stubborn b ignores new in-edges, so $\Delta c_s = 0$).

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Problems

P1 For which a, b, d is $\Delta c_s > 0$ when (a, b, d) is applied?

P2 Can *topological* properties of a, b, d guarantee $\Delta c_s > 0$ for *all* edge weights and stubbornness values?

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How does one modification change c_s?

Notations: $P = (I - \beta)W$, $F = (I - P)^{-1} = \sum_{k \geq 0} P^k$, $\mathbf{u} = w(1 - \beta_b)e_b$, $\mathbf{v} = e_d - e_a$.

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3. **Rank-one update:** $\hat{W} = W + w\, e_b(e_a - e_d)^T \Rightarrow \hat{P} = P - \mathbf{u}\mathbf{v}^T$. So $I - \hat{P} = (I - P) + \mathbf{u}\mathbf{v}^T$.

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$$\mathbf{x}_f = \underbrace{(I - (I - \beta)W)^{-1}\beta}_{P}\mathbf{x}(0)$$

2. Influence centrality: $c_s = \dfrac{\beta_s \mathbb{1}_n^T F e_s}{n}$.
3. **Rank-one update:** $\hat{W} = W + w\, e_b(e_a - e_d)^T \Rightarrow \hat{P} = P - \mathbf{u}\mathbf{v}^T$. So $I - \hat{P} = (I - P) + \mathbf{u}\mathbf{v}^T$.
4. **Sherman–Morrison formula:** $(I - P + \mathbf{u}\mathbf{v}^T)^{-1} = F - \dfrac{F\mathbf{u}\mathbf{v}^T F}{1 + \mathbf{v}^T F \mathbf{u}}$.

How does one modification change c_s?

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5. **Change in influence:** $\Delta c_s = \hat{c}_s - c_s = \dfrac{\beta_s}{n}\mathbb{1}_n^T(\hat{F} - F)e_s$, hence,

$$\Delta c_s = -\beta_s \frac{(\mathbb{1}_n^T F \mathbf{u})(\mathbf{v}^T F e_s)}{n(1 + \mathbf{v}^T F \mathbf{u})}$$

How does one modification change c_s? (contd.)

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$$\Delta c_s = -\beta_s \frac{(\mathbb{1}_n^T F \mathbf{u})(\mathbf{v}^T F e_s)}{n(1 + \mathbf{v}^T F \mathbf{u})}$$

- $\Delta c_s = 0$ if $\beta_b = 1$ (Assumption 2),
- $\mathbb{1}_n^T F \mathbf{u} > 0$ ($F \geq 0$ and $\beta_b < 1$),
- $1 + \mathbf{v}^T F \mathbf{u} > 0$ ($\det(I - P + uv) = \det(I - P)(1 + v(I - P)u)$),
- sign of Δc_s = sign of $(-\mathbf{v}^T F e_s)$ = sign of $(e_a - e_d)^T F e_s$.

The role of connectivity

Theorem

Let Assumptions 1 and 2 hold. The modification (a, b, d) increases c_s of a stubborn agent s if and only if

$$(e_a - e_d)^T \sum_{k=0}^{\infty} P^k e_s > 0.$$

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- So c_s increases iff the walks from s to a outweigh those from s to d; the choice of b is not important (only $\beta_b < 1$).

But:

- Verification of all walks and all weights: **expensive**.
- It depends on edge weights and stubbornness, which are hard to estimate.

$\Rightarrow$ **Solution:** We need to find conditions that depend only on topology.

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Locally Topologically Persuasive (LTP) agents

Definition (LTP agents)

$p \in \mathcal{V}$ is an **LTP** agent if

1. there is a non-stubborn $q \neq p$ such that every path from every stubborn agent to q contains p (the set of all such agents is $\mathcal{N}_p$),
2. if p is non-stubborn, no node c lies on all paths from the stubborn agents to p.

The non-influential agent q is then said to be **persuaded by** p.

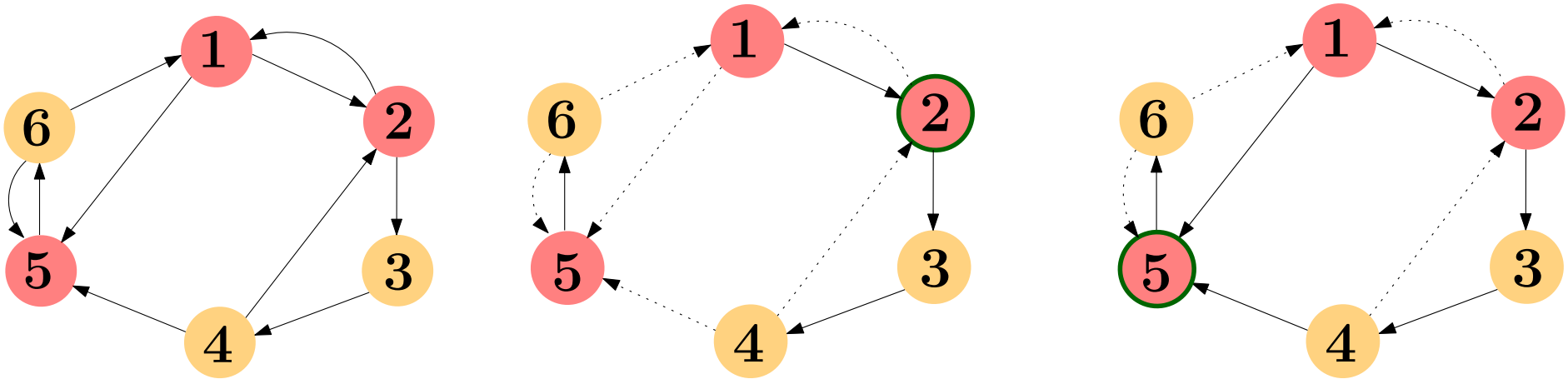
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(a) Network $\mathcal{G}$ **(b)** Agent 2 is LTP **(c)** Agent 5 is LTP

Figure: LTP agents

Endorsers and non-endorsers

Definition (Endorsers)

If a stubborn agent s is LTP, the **endorsers** of s are s itself and the agents in $\mathcal{N}_s$. All other agents are **non-endorsers** of s.

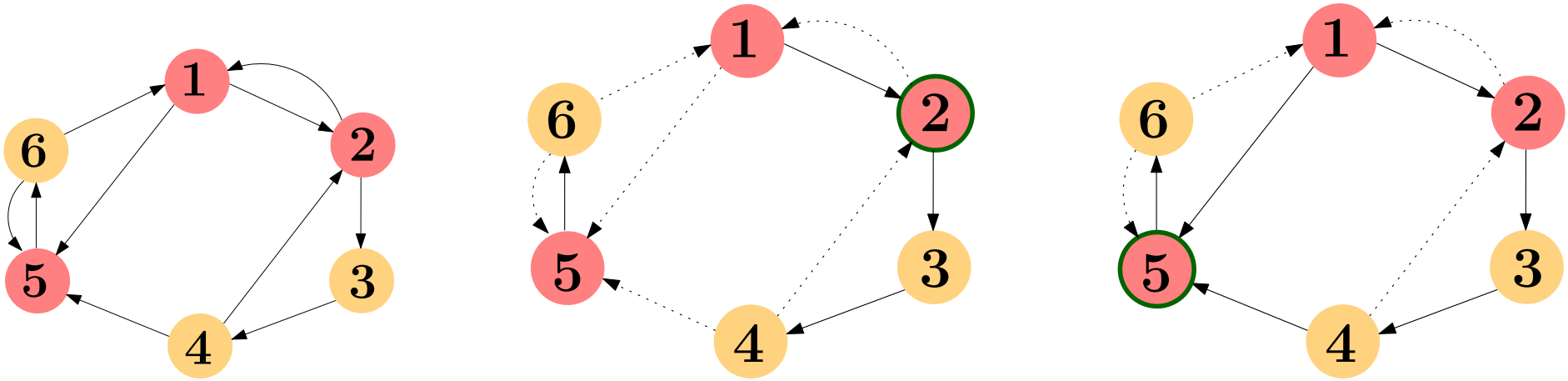
Endorsers and non-endorsers

Definition (Endorsers)

If a stubborn agent s is LTP, the **endorsers** of s are s itself and the agents in $\mathcal{N}_s$. All other agents are **non-endorsers** of s.

- If s is not LTP, its only endorser is s.
- With $m > 1$ stubborn agents, every stubborn agent has at least one endorser (itself) and at least one non-endorser.
- Non-stubborn endorsers: *supporters* (fan pages) of an influential individual.

Example: LTP agents and endorsers



(a) Network $\mathcal{G}$ **(b)** Agent 2 is LTP **(c)** Agent 5 is LTP

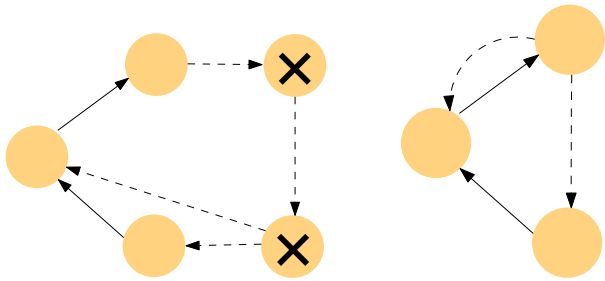
Figure: Stubborn LTP agents (red) and their endorsers.

- Endorsers of 2: $\{2, 3, 4\}$; the rest are non-endorsers.
- Endorsers of 5: $\{5, 6\}$.
- Do they have any interesting properties?

Kron Reduction Preliminaries

Kron reduction

- Given a Laplacian Q with digraph $G(Q)$:
  - α: nodes to be retained
  - reduced graph G_α of Q/α^c
- Schur complement to determine Q/α^c
  $(Q/\alpha^c = Q[\alpha] - Q[\alpha, \alpha^c]\, Q[\alpha^c]^{-1} Q[\alpha^c, \alpha])$



(a) $G(Q)$ **(b)** G_α

Figure: Nodes with crosses lie in α^c.

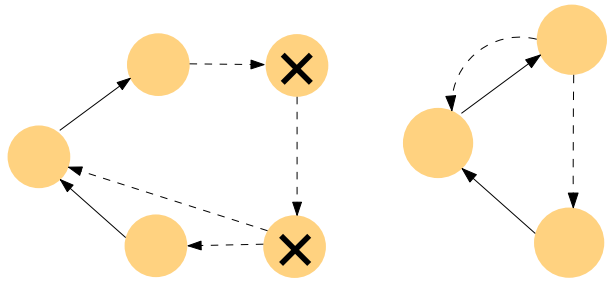
[9] F. Dorfler and F. Bullo, "Kron reduction of graphs with applications to electrical networks," IEEE Transactions on Circuits and Systems I: Regular Papers, vol. 60, no. 1, pp. 150–163, 2012

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Kron reduction of digraphs[8,9]

1. The Kron-reduced matrix Q/α^c (i.e. $Q[\alpha^c]^{-1}$ exists) is well-defined if every node $i \in \alpha^c$ has a path in $G(Q)$ from a node $j \in \alpha$.
2. If Q is a loopless (loopy) Laplacian matrix, then the Kron reduced matrix Q/α^c is also a loopless (loopy) Laplacian matrix.
3. An edge (i, j) exists in G_α if and only if there is a path i to j in $G(Q)$ such that all the nodes comprising the path belong to $\{i, j\} \cup \alpha^c$.

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Steady-State Behaviours of LTP nodes

- Under the FJ model, the final opinions of agents satisfy the equations:

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where $\mathbf{x}_s(0) = [x_{s_1}, ..., x_{s_m}]$ denote initial opinions of m stubborn agents.

- D is a Laplacian matrix.
- Using Kron reduction, we obtain a reduced form as:

$$D/\alpha^c \cdot z[\alpha] = \mathbf{0}$$

- Interestingly, D/α^c is also a Laplacian matrix.

Steady-State Behaviours of LTP nodes (contd.)

- Let p be LTP and $q \in \mathcal{N}_p$.
- Stubborn agents: $S_1, S_2, \cdots S_m$.
- $\alpha = \{p, q, S_1, S_2, \cdots, S_m\}$.

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- Steady state behaviour after Kron reduction:

$$\begin{bmatrix} d^1_{1,1} & d^1_{1,2} & d^1_{1,3} & \dots & d^1_{1,m+2} \\ d^1_{2,1} & d^1_{2,2} & 0 & \dots & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}^*_p \\ \mathbf{x}^*_q \\ \mathbf{x}_s(0) \end{bmatrix} = \mathbf{0}$$

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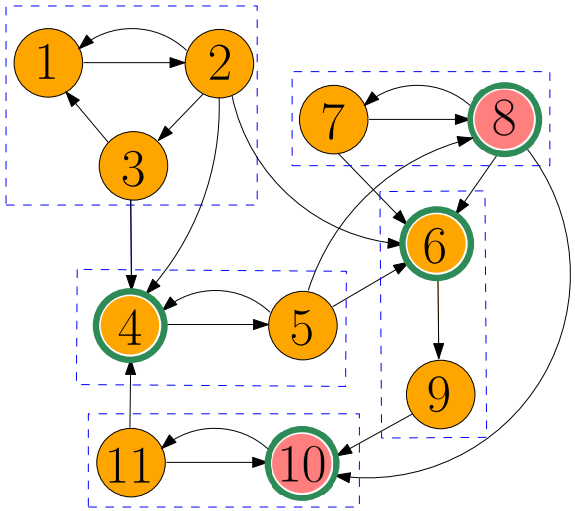
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- For agent q: $-d_{21}x^*_p + d_{22}x^*_q = 0 \implies x^*_p = x^*_q$.

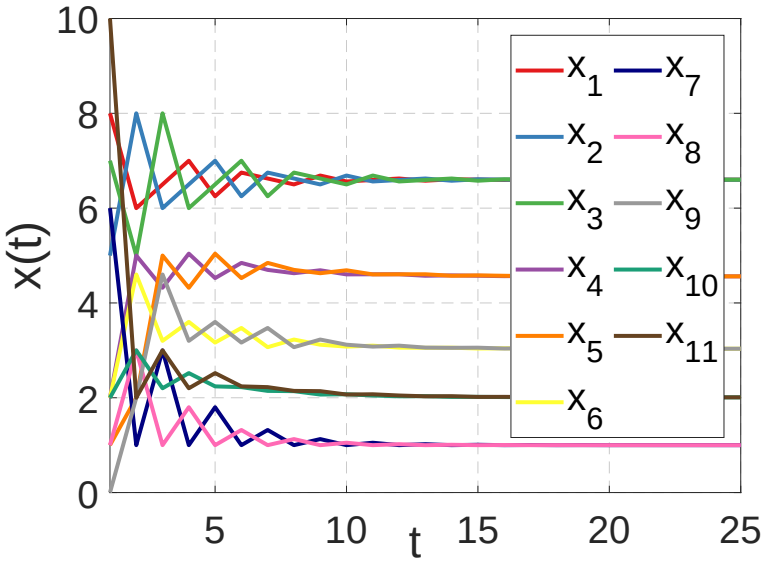
Key result

Consider a network $\mathcal{G}$ of n agents with m stubborn agents. The opinions of agents in G are governed by the FJ model. Then, the final opinions of an LTP agent p and the nodes in set N_p have the same steady state opinions.

Steady-State Behaviours of LTP nodes (contd.)



(a) Network $\mathcal{G}$



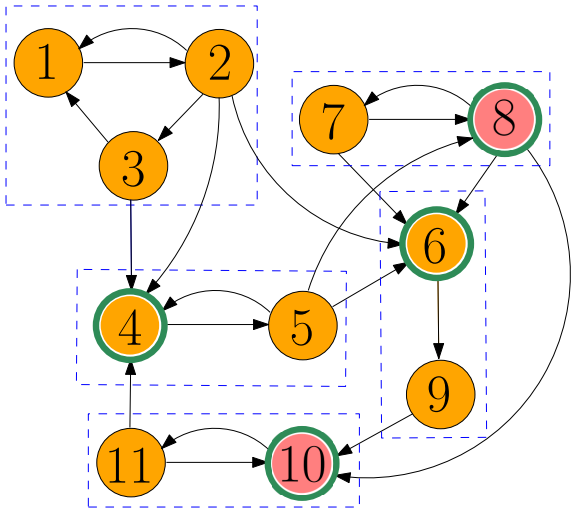
(b) Formation of desired opinion clusters

Figure: Suitable design of $\mathcal{G}$ for desired opinion clusters.

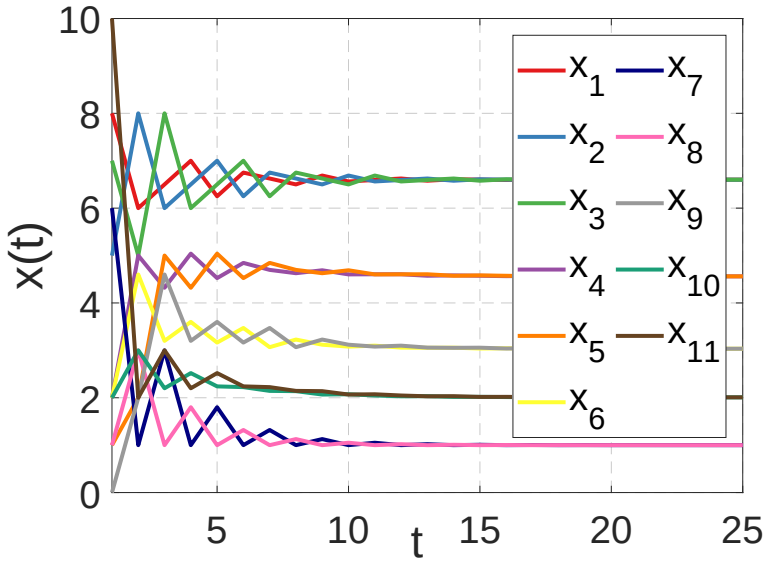
LTP agents

- Stubborn agents: 7 and 10,
- LTP agents: 1, 4, 7 and 10.
- Here, the stubborn agents directly control some of the agents!

Steady-State Behaviours of LTP nodes (contd.)



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LTP agents

- Stubborn agents: 7 and 10,
- LTP agents: 1, 4, 7 and 10.
- Here, the stubborn agents directly control some of the agents! Can we leverage this information for influence enhancement?

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From FJ steady state to a graph $G(R)$ using a Laplacian R

The Laplacian matrix R

- At steady state:
  - $\mathbf{x}^* = (I - \beta)W\mathbf{x}^* + \beta\mathbf{x}(0)$
  - $\bar{x} = \mathbb{1}_n^T \mathbf{x}^*/n$

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- Stacking them gives:

$$\underbrace{\begin{bmatrix} I - (I-\beta)W & -\beta[[n],[m]] & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbb{1}_n^T/n & \mathbf{0} & 1 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} \mathbf{x}^* \\ \mathbf{x}_s(0) \\ \bar{x} \end{bmatrix}}_{\mathbf{z}} = \mathbf{0}$$

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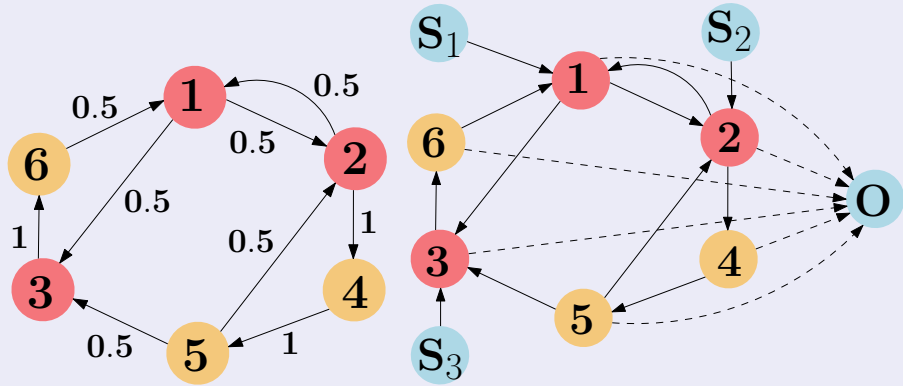
such that:

- $R\mathbb{1} = \mathbf{0}$, diagonal entries ≥ 0, off-diagonal entries ≤ 0
- R is a Laplacian matrix!

From FJ steady state to a graph $G(R)$ using a Laplacian R (contd.)

Construction of $G(R)$

- R is a **Laplacian** matrix.
- $G(R)$ has $n + m + 1$ nodes:
  - the n agents,
  - the sources S (initial opinion of the stubborn agents),
  - the sink O ($\bar{x}$).
- Edges:
  - $S_j \to j$,
  - every agent $\to O$,
  - an edge (j, i) of $\mathcal{G}$ stays an edge of $G(R)$ if $\beta_i < 1$.



(a) Network $\mathcal{G}$ **(b)** Network $G(R)$

Figure: Illustration of $\mathcal{G}$ and $G(R)$

Kron reduction of $G(R)$ yields influence centrality

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Lemma (Influence centrality as an edge weight)

For $\alpha = \{S_1, \ldots, S_m, O\}$*:*

$$R/\alpha^c = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\frac{1}{n}\mathbb{1}_n^T(I-(I-\beta)W)^{-1}\beta[[n],[m]] & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -[c_1, \ldots, c_m] & 1 \end{bmatrix}$$

so the edge (S_j, O) *in* G_α *has weight* $-\mathbf{c_j}$.

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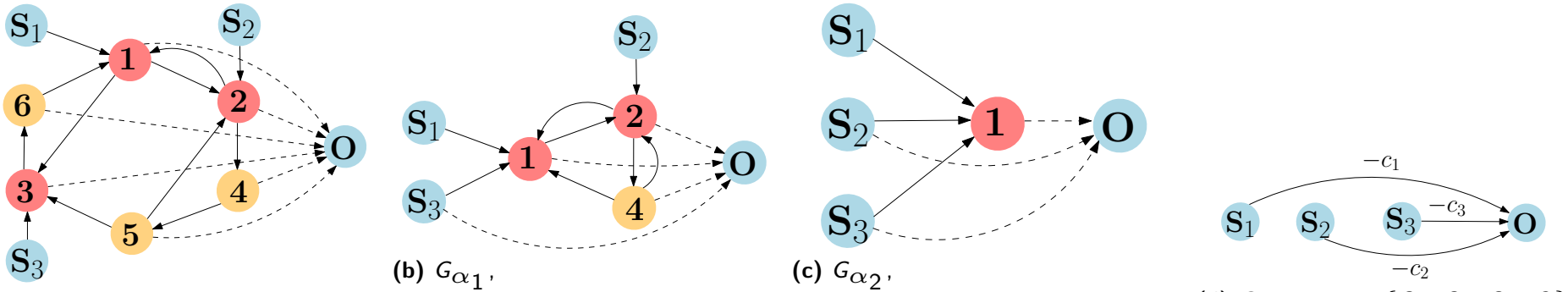
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Kron reduction is well behaved

Let $\alpha \supseteq \{S_1, \ldots, S_m, O\}$. Then R/α^c is a well-defined *loopless Laplacian*; this persists under iterative reduction.

Example: Kron reduction of $G(R)$



(a) Original $G(R)$

(b) G_{α_1}, $\alpha_1 = \{1, 2, 4, S_1, S_2, S_3, O\}$

(c) G_{α_2}, $\alpha_2 = \{1, S_1, S_2, S_3, O\}$

(d) G_{α_3}, $\alpha_3 = \{S_1, S_2, S_3, O\}$

Figure: Iterative Kron reduction of $G(R)$ (self-loops omitted); the last graph has edges (S_j, O) of weight $-c_j$.

- The first step removes nodes $(\{3, 5, 6\}$, then $\{2, 4\}$, then $\{1\})$, so the effect of a local change can be tracked in a much smaller graph.
- $\alpha \supseteq \{S_1, \ldots, S_m, O\}$ is required: G_{α_3} cannot be reduced any further.

How does a changed edge propagates to the reduced graph?

Theorem

Let Q be a loopy Laplacian, Q/α^c well defined, and let the weight of edge (u, v) in $G(Q)$ change. Then in G_α (with $j, k \in \alpha$):

(i) *$u, v \in \alpha$: only the edge (u, v) changes.*

(ii) *$u \in \alpha$, $v \in \alpha^c$: edge (u, j) changes if there is a path $u \to j$ through (u, v) with all interior nodes in α^c.*

(iii) *$u \in \alpha^c$, $v \in \alpha$: edge (k, v) changes if there is a path $k \to v$ through (u, v) with interior nodes in α^c.*

(iv) *$u, v \in \alpha^c$: edge (j, k) changes if there is a path $j \to k$ through (u, v) with interior nodes in α^c.*

and the reduced weight moves in the same direction as the original edge.

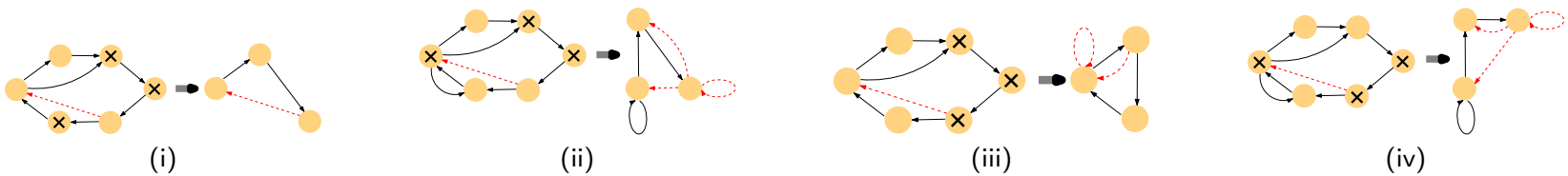


Figure: Left: $G(Q)$ (crosses $= \alpha^c$); right: G_α. Perturbed edges are dashed red.

Recap: the role of connectivity

Theorem

Let Assumptions 1 and 2 hold. The modification (a, b, d) increases c_s of a stubborn agent s if and only if

$$(e_a - e_d)^T \sum_{k=0}^{\infty} P^k e_s > 0.$$

- $e_a^T \sum_k P^k e_s$ = aggregate weight of all walks from s to a in $G(P)$.
- So c_s increases iff the walks from s to a outweigh those from s to d; the choice of b is not important (only $\beta_b < 1$).

But:

- Verification of all walks and all weights: **expensive**.
- It depends on edge weights and stubbornness, which are hard to estimate.

$\Rightarrow$ We need to find conditions that depend only on topology.

Redundant edge modifications

Theorem

Let p be an LTP agent and Assumptions 1 and 2 hold. If $a, d \in \mathcal{N}_p \cup \{p\}$, the modification (a, b, d) is **redundant***: the influence centrality of every agent is unchanged, for all weights and stubbornness values.*

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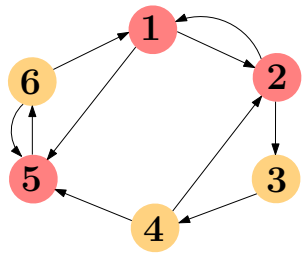
- If a, d are both endorsers of s, then (a, b, d) is redundant, so adding an edge *from the desired stubborn agent can fail* to increase its influence.

Redundant edge modifications

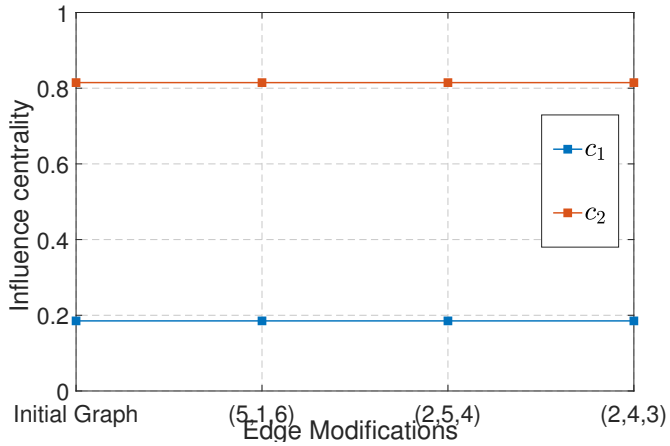
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- If a, d are both endorsers of s, then (a, b, d) is redundant, so adding an edge *from the desired stubborn agent can fail* to increase its influence.
- Example: $\mathcal{N}_2 = \{3, 4\}$ and $\mathcal{N}_5 = \{6\}$ give the redundant modifications $(5, 1, 6)$, $(2, 5, 4)$ and $(2, 4, 3)$.



(a) Network $\mathcal{G}$



(b) Redundant modifications in $\mathcal{G}$

Figure: Original network (red: stubborn agents) and the influence centralities under the three modifications.

Edge modifications to increase influence

Theorem

Let Assumptions 1 and 2 hold. The modification (a, b, d) **increases** *the influence centrality of stubborn agent* s *if*

- a *is an* **endorser** *of* s*, and*
- d *is a* **non-endorser** *of* s*.*

This holds for any edge weights and stubbornness values.

Edge modifications to increase influence

Theorem

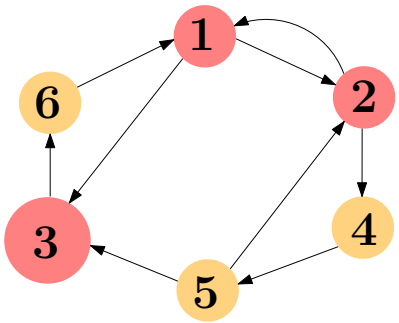
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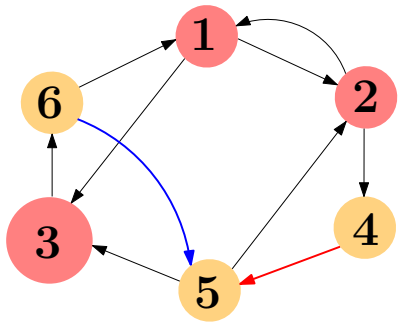
This holds for any edge weights and stubbornness values.

- Selection depends *only on topology*.
- Edges may be added from s *or* from its non-stubborn endorsers (fan pages): a larger class than edges from s alone.
- Since $\mathbf{c}^T \mathbb{1} = 1$, influence gained by s is lost by the other stubborn agents.

Example: an effective modification



(a) Network $\mathcal{G}$



(b) Edge modification (3, 1, 6)

Here $a = 3$ is an endorser of the desired stubborn agent 3 and $d = 6$ is a non-endorser. By the closed form,

$$\Delta c_s = -\beta_s \frac{\overbrace{(\mathbb{1}_n^T F \mathbf{u})}^{\geq 0} (\mathbf{v}^T F e_s)}{n \underbrace{(1 + \mathbf{v}^T F \mathbf{u})}_{>0}},$$

and $\Delta c_s > 0 \iff F_{a,s} > F_{d,s}$: the endorser a is better connected to s than d.

Key advantages of the topology-based approach

- Applies to **any desired stubborn agent** (not only an external influencer) and to **any weakly connected digraph**; compare with Wang et al., "Adding links wisely: how an influencer seeks for leadership in opinion dynamics?", arXiv:2506.12463.
- Redundant and effective modifications are identified **regardless of edge weights and stubbornness values**.
- Identifying the LTP agents and endorsers costs only $O\big(m(n+|\mathcal{E}|)\big)$, with $m \ll n$ stubborn agents; while exploring all the walks needs F, i.e. $O(kn^2)$.

Outline

1 Introduction and Motivation

2 Problem Formulation: Influence Enhancement Through Edge Modifications

3 Impact of an Edge Modification

4 Topology-Based Edge Modifications

5 Kron Reduction Framework for Influence Enhancement

6 Conclusions

Conclusion

Key insights

- **Exact criterion**: (a, b, d) raises c_s iff s reaches a with more walk weight than d; closed form for Δc_s.
- **Kron reduction of** $G(R)$ turns influence centrality into an edge weight and tracks how a local change propagates (Theorem 2).
- **Topology-only guarantees**: redundant modifications (both nodes behind the same LTP agent) and always-effective ones (endorser $\rightarrow$ non-endorser), independent of weights and stubbornness.

Future directions

- Applying these results to other models of opinion formation.
- Developing an influence maximisation algorithms, and richer models of recommendation acceptance.

Thank you